Name:

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MWF 9 Tina Torkaman
MWF 10:30 Jameel Al-Aidroos
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MWF 12 Stepan Paul
MWF 12 Hunter Spink
MWF 12 Nathan Yang
MWF 1:30 Fabian Gundlach
MWF 1:30 Flor Orosz-Hunziker
MWF 3 Waqar Ali-Shah

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and nice unless stated otherwise.
- **Show your work**. Except for problems 1-3 and 6, we need to see details of your computation. If you are using a theorem for example, state the theorem.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

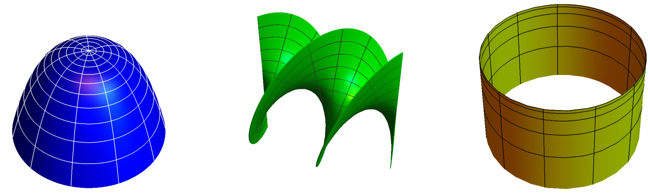
Problem 1) True/False questions (20 points). No justifications are needed.

1)	T	F	The length of the vector $\vec{v} = [3,4,5]$ is equal to the distance from the point $(1,1,1)$ to the point $(-2,-3,-4)$.
2)	T	F	There is a non-constant vector field $\vec{F}(x,y,z)$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{F})$.
3)	T	F	For every $\vec{v}$ and $\vec{w}$, the projection of $\vec{v}$ onto $\vec{w}$ always has the same length as the projection of $\vec{w}$ onto $\vec{v}$.
4)	T	F	If $\vec{F}(x,y,z) = [\sin(z), \cos(z), 0]$, then $\text{curl}(\text{curl}(\text{curl}(\vec{F}))) = \vec{F}$.
5)	T	F	If S is the graph $z = x^6 + y^6$ above $x^2 + y^2 \leq 1$ oriented upwards and $\vec{F}(x,y,z) = [0,0,z]$, then the flux of $\vec{F}$ through S is positive.
6)	T	F	If S is the unit sphere oriented outwards and $\vec{F}(x,y,z) = [0,0,z^2]$, then the flux of $\vec{F}$ through the upper hemisphere of S is the same as the flux through the lower hemisphere.
7)	T	F	If the divergence of $\vec{F}$ is zero, then $\vec{F}$ is a gradient vector field.
8)	T	F	If E is the unit ball $x^2+y^2+z^2 \leq 1$ and $\vec{F}$ is the curl of some other vector field, then $\int\int\int_E \text{div}(\vec{F})dV = 4\pi/3$.
9)	T	F	The curve $\vec{r}(t) = [t,t]$ is a flow line of $\vec{F}(x,y) = [x, 2y]$.
10)	T	F	The integral $\iint_R \sqrt{1+f(u,v)^2}\ dudv$ is the surface area of the surface parametrized by $\vec{r}(u,v) = [u,v,f(u,v)]$ for $(u,v) \in R$.
11)	T	F	The volume of a parallelepiped with corners $A, B = A+\vec{v}, C = A+\vec{w}, D = A+\vec{v}+\vec{w}$ and $A+\vec{u}, B+\vec{u}, C+\vec{u}, D+\vec{u}$ is $\lvert\vec{u}\cdot(\vec{v}\times\vec{w})\rvert$.
12)	T	F	The arc length of the curve $\vec{r}(t) = [2\cos(t), 2\sin(t), 0]$ with $0 \leq t \leq 2\pi$ is 4π.
13)	T	F	If $\vec{F}$ and $\vec{G}$ are two vector fields for which the divergence is the same, then $\vec{F}-\vec{G}$ is a constant vector field.
14)	T	F	If $\vec{F}, \vec{G}$ are two vector fields which have the same curl, then $\vec{F}-\vec{G}$ is irrotational.
15)	T	F	The parametrization $\vec{r}(u,v) = [1+u, v, u+v]$ describes a plane.
16)	T	F	Any function $u(x,y)$ that obeys the partial differential equation $u_x + u_y - u_{xx} = 1$ has no local minima.
17)	T	F	If $\vec{F} = [P,Q,R]$ is a vector field so that $[P_x, Q_y, R_z] = [0,0,0]$, then it is incompressible meaning that the divergence is zero everywhere.
18)	T	F	If $f(x,g(x)) = 0$, then $g'(x) = -f_x/f_y$ provided f_y is not zero.
19)	T	F	The equation $x^2 - (y-1)^2 + z^2 + 2z = -1$ represents a two-sheeted hyperboloid.
20)	T	F	If $\vec{F}(\vec{r}(u,v)) = (\vec{r}_u \times \vec{r}_v)/\lvert\vec{r}_u \times \vec{r}_v\rvert$, then the absolute value of the flux of $\vec{F}$ through a closed bounded surface S parametrized by $\vec{r}(u,v)$ is the surface area of S.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

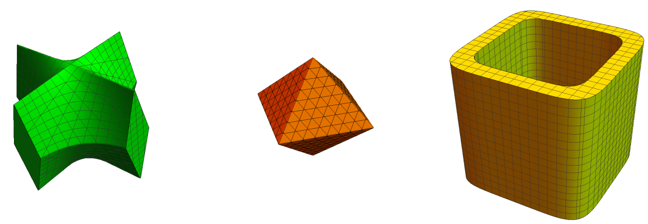
A B C



Parametrized surface $\vec{r}(u,v)$	A-C
$[u\sin(v), u\cos(v), -u^3]$	
$[u, v\sin(u), v\cos(u)]$	
$[\sin(u), \cos(u), -v^3]$	

b) (2 points) Match the solids. There is an exact match.

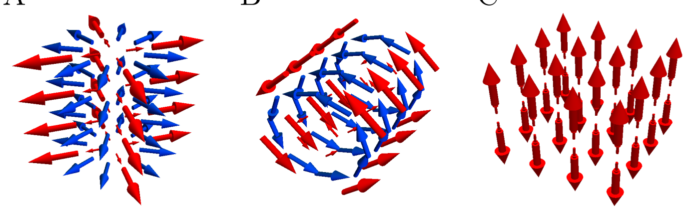
A B C



Solid	A-C
$1 < x^6 + y^6 < 2$	
$\|x\|\|y\| + \|z\| < 1$	
$\|x\| + \|y\| + \|z\| < 3$	

c) (2 points) The figures display vector fields $\vec{F}$. Match them.

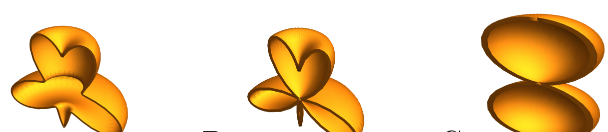
A B C



Field	A-C
$\vec{F}(x,y,z) = [0,0,z]$	
$\vec{F}(x,y,z) = [-z,0,x]$	
$\vec{F}(x,y,z) = [x,y,0]$	

d) (2 points) Match the spherical plots.

Surface	A-C
$\rho(\theta,\phi) = 1 + \sin(4\phi)$	
$\rho(\theta,\phi) = 2 + \sin(4\phi)$	
$\rho(\theta,\phi) = 1 + \cos(2\phi)$	

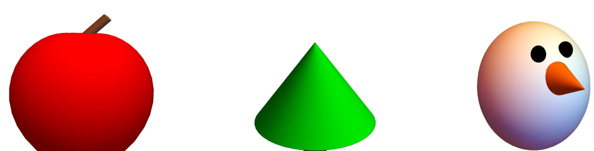


A B C

e) (1 point) Name a partial differential equation (PDE) for a function $u(t,x)$ discussed in this course which involves a term uu_x.

f) (1 point) Match each surface S to a graphic that contains S.

Surface S	A-C
$r^2 - (1-z)^2 = 0$	
$x^2 + y^2 + z^2 = 1$	
$\rho(\theta,\phi) = \sin^2(\phi/2)\phi$	



A B C

Problem 3) (10 points)

a) (3 points) Ed Sheeran's **"Shape of You"**, released in January this year, has been a critical success: it peaked at number-one on the singles charts of 44 countries and is currently the most streamed song on **Spotify.** But what is the shape of you? Which of the letters are not simply connected (SC)?

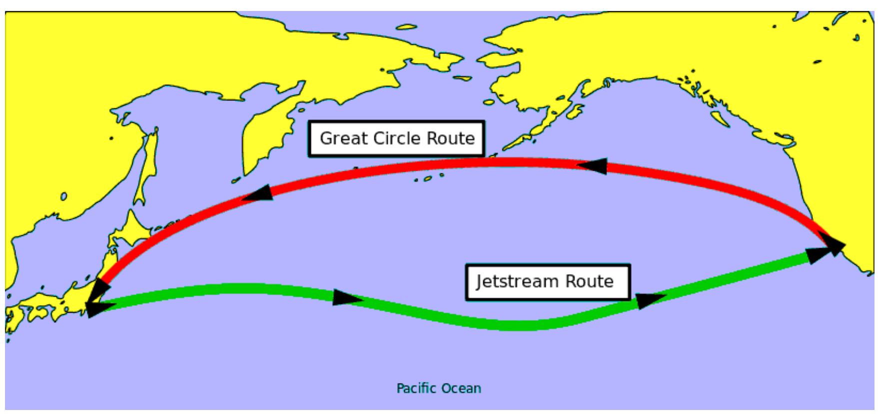
	Check if not SC
S	
H	
A	
P	
E	

	Check if not SC
O	
F	
Y	
O	
U	

b) (4 points)

A plane flies from **Los Angeles** to **Tokyo** along the great circle route A and comes back via the jet stream route B. There is a force field $\vec{F}$ acting on the plane so that the work along A is $\int_A \vec{F} \cdot d\vec{r}$ and the work along B is $\int_B \vec{F} \cdot d\vec{r}$. You know that there is a potential function f such that $\vec{F} = \nabla f$. Check the statements that must be true.

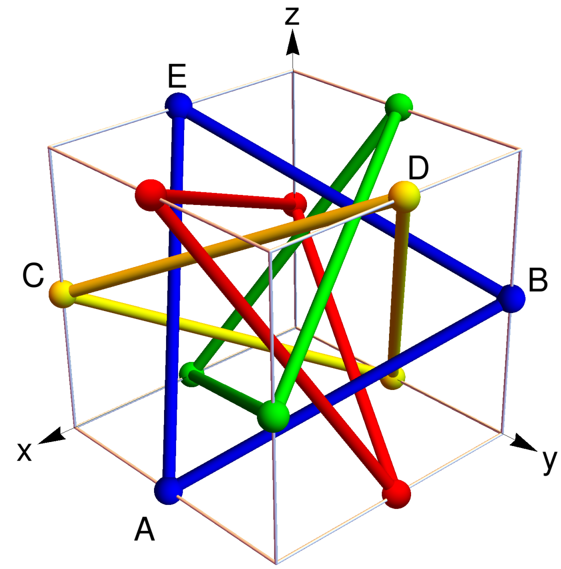
$\int_A \vec{F} \cdot d\vec{r} = 0$	
$\int_B \vec{F} \cdot d\vec{r} = 0$	
$\int_A \vec{F} \cdot d\vec{r} + \int_B \vec{F} \cdot d\vec{r} = 0$	
$\int_A \vec{F} \cdot d\vec{r} - \int_B \vec{F} \cdot d\vec{r} = 0$	



c) (3 points) Let E be the solid given by $x^8 + y^8 + z^8 \leq 4, y \geq 0$. Let S be the boundary of E with outward orientation. Consider the vector fields $\vec{F} = [x, y, z]$, $\vec{G} = [x, y, -z]$ and $\vec{H} = [x + y, y^2 + z^2, yz]$. Check the correct box in each line:

Flux integral	$<$ Vol(E)	$=$ Vol(E)	$>$ Vol(E)
$\iint_S \vec{F} \cdot d\vec{S}$			
$\iint_S \vec{G} \cdot d\vec{S}$			
$\iint_S \vec{H} \cdot d\vec{S}$			

Problem 4) (10 points)



In topology one knows the **Danzer cube**. It is an example of what one calls a "non-shellable triangulation" of the cube. The picture shows four of the triangles.

a) (5 points) Find the distance between the line joining

$$A = (2, 1, 0) \text{ and } B = (0, 2, 1)$$

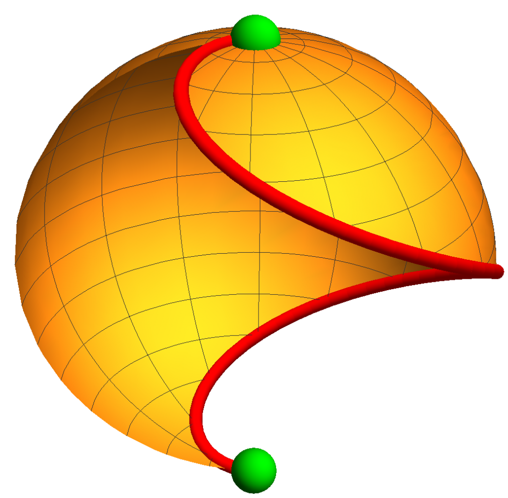
and the line joining

$$C = (2, 0, 1) \text{ and } D = (1, 2, 2) .$$

b) (5 points) Find the area of the triangle ABE, where

$$E = (1, 0, 2) .$$

Problem 5) (10 points)



a) (6 points) Find the **surface area** of

$$\vec{r}(t, s) = [\cos(t)\sin(s), \sin(t)\sin(s), \cos(s)]$$

$0 \leq t \leq 2\pi, 0 \leq s \leq t/2$.

b) (4 points) The part of the boundary curve when $s = t/2$ is defined as

$$\vec{r}(t) = [\cos(t)\sin(t/2), \sin(t)\sin(t/2), \cos(t/2)] .$$

Compute the number $|\int_0^{2\pi} \vec{r}\,'(t)\ dt|$.

Problem 6) (10 points)

The **Longy School of Music** at 27 Garden Street shows an abstract art work featuring a sphere, a cone and a cylinder. You build a model. Your parametrization should use the variables provided. No further justifications are needed in this problem.

a) (2 points) Parametrize the sphere $x^2 + y^2 + (z-5)^2 = 9$.

$\vec{r}(\theta, \phi) = [\quad , \quad , \quad]$

b) (3 points) Parametrize the cylinder $x^2 + y^2 = 4$.

$\vec{r}(\theta, z) = [\quad , \quad , \quad]$

c) (3 points) Parametrize the cone $4y^2 + 4(z-5)^2 = x^2$.

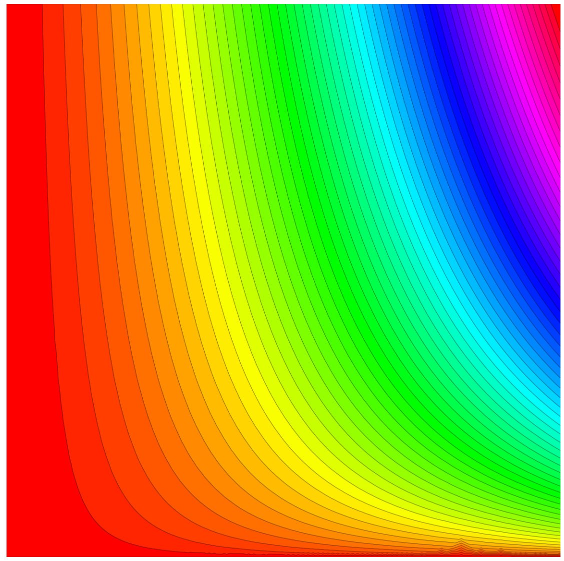
$\vec{r}(\theta, x) = [\quad , \quad , \quad]$

d) (2 points) Parametrize the grass floor $z = \sin(99x + 99y)$.

$\vec{r}(x, y) = [\quad , \quad , \quad]$

Photo: O. Knill, December 2017

Problem 7) (10 points)



a) (6 points) Find the **linearization** $L(x, y)$ of

$$f(x, y) = \sqrt{x^3 y}$$

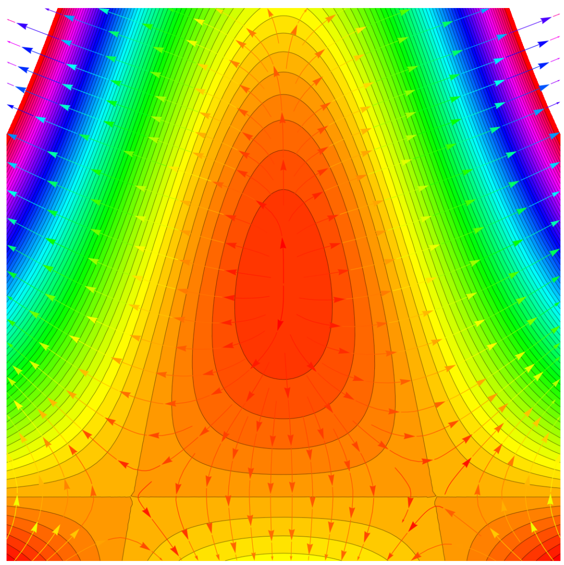
at $(x_0, y_0) = (10, 1000)$.

b) (4 points) Estimate the value

$$\sqrt{11^3 \cdot 999}$$

using the linearization in a).

Problem 8) (10 points)



The vector field $\vec{F}$ is a **gradient field** with potential function

$$f(x, y) = y^2 + 4yx^2 + 4x^2 \ .$$

a) (8 points) Find and classify all the critical points of f.

b) (2 points) Does f have a global maximum or global minimum on the whole xy-plane? Only a brief explanation is needed. Some context: the critical points of f are the equilibrium points of $\vec{F}$. A critical point is called a **sink** if all vectors nearby point towards it. **Sinks** correspond to maxima of f. The minima of f are also called **source** as all vectors points nearby point away of it. An equilibrium is called **hyperbolic** if there are vectors pointing both away and towards it. These are the saddle points of f.

Problem 9) (10 points)

The top tower of the Harvard **Memorial Hall** is a **square frustum** of height $h = 9$. On the **Moscow Papyrus** written in 1850 BC, the volume of such a truncated square pyramid with side lengths x, y of the top and bottom faces, has already been given with the formula $h(x^2 + xy + y^2)/3$. Using this almost four-millennia year old formula and Lagrange, find the minimal volume

$$f(x, y) = 3x^2 + 3xy + 3y^2$$

under the constraint

$$g(x, y) = 3x + 2y = 14 .$$

You don't have to justify whether the solution is a minimum.

Photo: O. Knill, November 2017

Problem 10) (10 points)

When properly aired, sand becomes **liquid sand** and you can take a bath in a sand tank. Assume the force field acting on a body floating in it is

$$\vec{F} = [-y, x, z] .$$

The flux $\int\int_S \vec{F} \cdot d\vec{S}$ of this vector field through the surface of the body is the uplift. What is the uplift of the football

$$x^2 + y^2 \leq \cos^2(z)$$

with $-\pi/2 \leq z \leq \pi/2$, with the outward orientation?

Source: Youtube, Mark Rober, November 28, 2017

Problem 11) (10 points)

Find the **flux** of the curl of the vector field

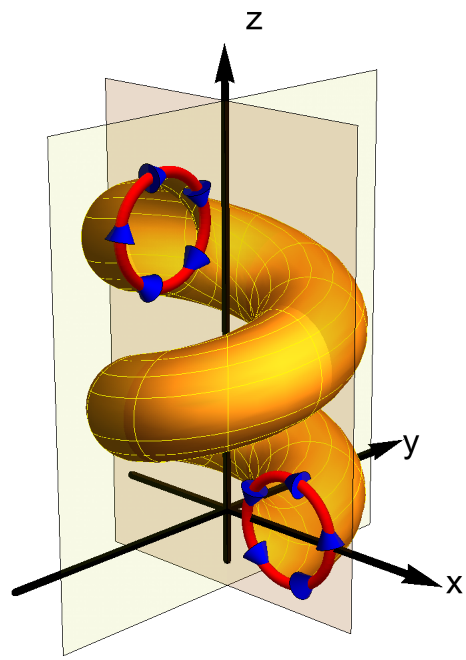
$$\vec{F}(x, y, z) = [-z, z + \sin(xyz), x - 3] + [x^5, y^7, z^4]$$

through the **twisted surface** oriented inwards and parametrized by

$$\vec{r}(t, s) = [(3+2\cos(t))\cos(s), (3+2\cos(t))\sin(s), s+2\sin(t)]$$

where $0 \leq s \leq 7\pi/2$ and $0 \leq t \leq 2\pi$.

Hint: This parametrization leads correctly already to a vector $\vec{r}_t \times \vec{r}_s$ pointing inwards. The boundary of the surface is made of two circles $\vec{r}(t, 0)$ and $\vec{r}(t, 7\pi/2)$. The picture gives the direction of the velocity vectors of these curves (which in each case might or might not be compatible with the orientation of the surface).



Problem 12) (10 points)

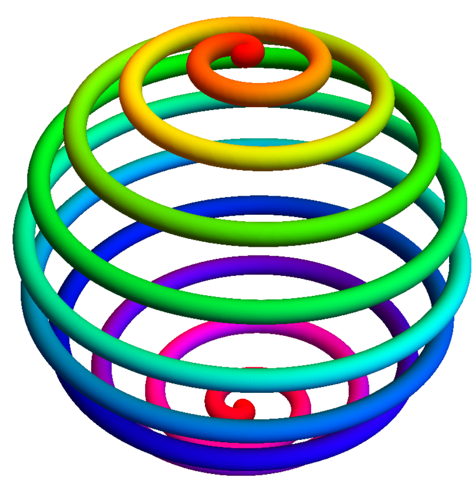
Find the line integral of the vector field

$$\vec{F}(x, y, z) = [yz + x^2, xz + y^2 + \sin(y), xy + \cos(z)]$$

along the **spherical curve**

$$\vec{r}(t) = [\cos(20t)\sin(t), \sin(20t)\sin(t), \cos(t)] ,$$

where $0 \leq t \leq \pi$.



Problem 13) (10 points)

Look at the shaded region G bounded by a circle of radius 2 and an inner **figure eight lemniscate** with parametric equation

$$\vec{r}(t) = [\sin(t), \sin(t)\cos(t)]$$

with $0 \leq t \leq 2\pi$. The picture shows the curve and the arrows indicate some of the velocity vectors of the curve. Find the area of this region G.