Name:

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MWF 12 Stepan Paul
MWF 12 Hunter Spink
MWF 12 Nathan Yang
MWF 1:30 Fabian Gundlach
MWF 1:30 Flor Orosz-Hunziker
MWF 3 Waqar Ali-Shah

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and nice unless stated otherwise.
- **Show your work**. Except for problems 1-3, we need to see details of your computation. If you are using a theorem for example, state the theorem.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

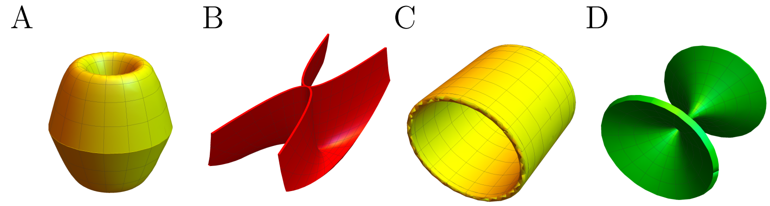
Problem 1) True/False questions (20 points). No justifications are needed.

1)	T	F	The surface $-x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid.
2)	T	F	The vector projection $\vec{\text{proj}}_{\vec{w}}(\vec{v})$ of a vector $\vec{v}$ onto a non-zero vector $\vec{w}$ is always non-zero.
3)	T	F	The linearization of $f(x, y) = 5 + 7x + 3y$ at any point (a, b) is the function $L(x, y) = 5 + 7x + 3y$.
4)	T	F	For any function $f(x, y, z)$, for any unit vector $\vec{u}$ and any point (x_0, y_0, z_0) we have $D_{\vec{u}} f(x_0, y_0, z_0) = -D_{-\vec{u}} f(x_0, y_0, z_0)$.
5)	T	F	There is a vector field $\vec{F} = [P, Q, R]$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{F})$.
6)	T	F	The formula $\lvert\vec{v} \times \vec{w}\rvert = \lvert\vec{v}\rvert\lvert\vec{w}\rvert \sin(\alpha)$ holds if $\vec{v}, \vec{w}$ are vectors in space and α is the angle between them.
7)	T	F	If $\vec{F} = [-2y, 2x]$ and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise, then $\int_C \vec{F} \cdot d\vec{r} = 16\pi$.
8)	T	F	The parametrization $\vec{r}(u, v) = [u, \sqrt{1 - u^2 - v^2}, v], u^2 + v^2 \leq 1$ describes a half sphere.
9)	T	F	The vectors $\vec{v} = [1, 0, 1]$ and $\vec{w} = [-1, 1, 1]$ are perpendicular.
10)	T	F	If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) then $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C.
11)	T	F	The vector field $\vec{F} = [e^x, e^y, e^z]$ satisfies $\text{grad}(\text{div}(\vec{F})) = \vec{F}$.
12)	T	F	If $\vec{F}(x, y, z)$ has zero curl everywhere in space, then $\vec{F}$ is a gradient field.
13)	T	F	If $\vec{r}(u, v)$ is a parametrization of the surface $g(x, y, z) = x^2 + e^{y^3} + z^2 = 5$ then for any u and v we have $\nabla g(\vec{r}(u, v)) \cdot \vec{r}_u(u, v) = 0$.
14)	T	F	The equation $\text{grad}(\text{div}(\text{grad}(f))) = [0, 0, 0]$ always holds.
15)	T	F	There is a non-constant function $f(x, y, z)$ such that $\text{grad}(f) = \text{curl}(\text{grad}(f))$ everywhere.
16)	T	F	If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S oriented outwards is the volume of the enclosed solid.
17)	T	F	If $f(x, y)$ is maximized at (a, b) under the constraint $g(x, y) = c$, then $\nabla f(a, b)$ and $\nabla g(a, b)$ are parallel.
18)	T	F	The distance between a point P and the line L through two different points A, B is given by the formula $\lvert\vec{PA} \times \vec{AB}\rvert / \lvert\vec{PA}\rvert$.
19)	T	F	The unit tangent vector $\vec{T}(t)$ is always perpendicular to the vector $\vec{T}\,'(t)$.
20)	T	F	The vector field $\vec{F}(x, y, z) = [x^5, x^6, x^7]$ can not be the curl of another vector field.

Problem 2) (10 points) No justifications are necessary.

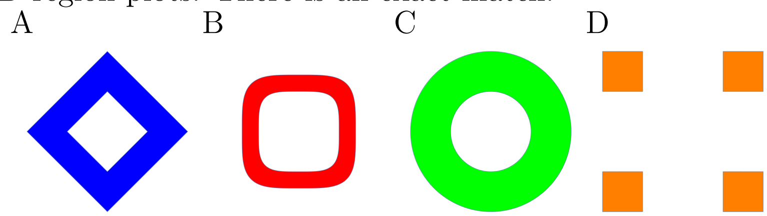
a) (2 points) Match the following surfaces. There is an exact match.

Surface $\vec{r}(u,v) =$	A-D
$[u, u^2 v, v^2]$	
$[u^2\cos(v), u, u^2\sin(v)]$	
$[\cos(v), \sin(u), \sin(v)]$	
$[v\cos(u), v\sin(u), \sin(v)]$	



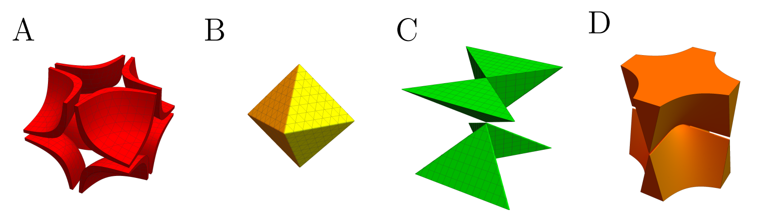
b) (2 points) Match the following 2D region plots. There is an exact match.

Region	A-D
$1 \leq x^2 \leq 4, 1 \leq y^2 \leq 4$	
$1 \leq x^2 + y^2 \leq 4$	
$1 \leq x^4 + y^4 \leq 4$	
$1 \leq \vert x\vert + \vert y\vert \leq 2$	



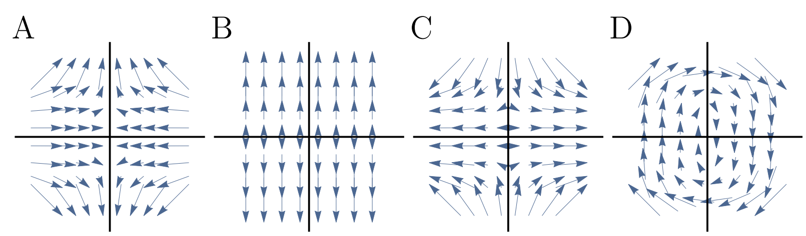
c) (2 points) Match the following 3D regions. There is an exact match.

Solid	A-D
$\vert x\vert + \vert y\vert + \vert z\vert \leq 2$	
$\vert x\vert < \vert y\vert < \vert z\vert$	
$1 < \vert xyz\vert < 2$	
$x^2y^2 < z^2$	



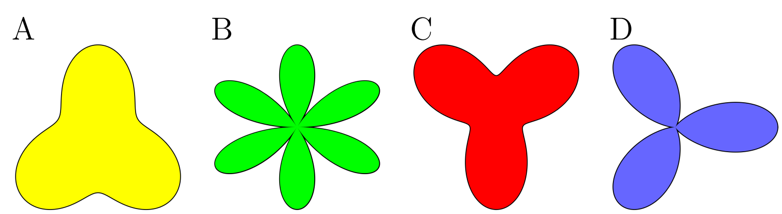
d) (2 points) The following figures display vector fields. There is an exact match.

Field	A-D
$\vec{F}(x,y) = [0, y]$	
$\vec{F}(x,y) = [-x, y^3]$	
$\vec{F}(x,y) = [y^3, -x]$	
$\vec{F}(x,y) = [x, -y^3]$	



e) (2 points) The following figures display polar regions. There is an exact match.

Polar region	A-D
$r \leq 1 + \cos(3\theta)$	
$r \leq 2 + \sin(3\theta)$	
$r \leq 3 - \sin(3\theta)$	
$r \leq \vert\sin(3\theta)\vert$	



Problem 3) (10 points)

a) (6 points)

The concept of **boundary** plays an important role in integral theorems. In each of the following six rows, check exactly one entry which best describes the boundary.

The boundary of	solid	surface	curves	points	empty
$x^2 + y^2 + z^2 = 1$					
$x^2 + y^2 = 1, z = 0$					
$x^2 + y^2 + z^2 \leq 1, x = y = 0$					
$x^2 + y^2 + z^2 \leq 1$					
$x^2 + y^2 \leq 1, z = 0$					
$x^2 + y^2 = 1, z^2 \leq 1$					

b) (4 points) Match the following partial differential equations (PDEs) by picking 4 from the 5 given choices A-E.

PDE	Enter A-E
heat equation	
wave equation	
transport equation	
Burgers equation	

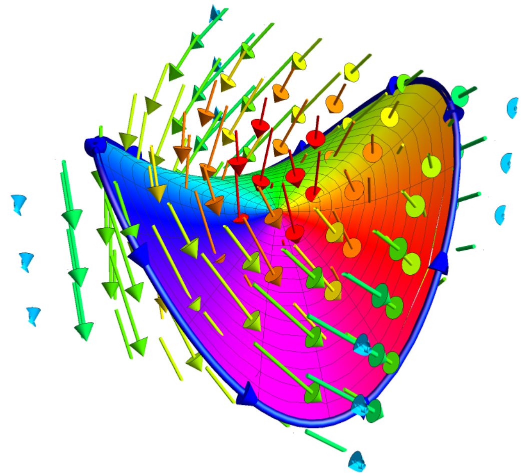
$u_{xx} = u_t$	A
$u_{xx} = u_t + uu_x$	B
$u_x = u_t$	C
$u_{xx} = -u_{tt}$	D
$u_{xx} = u_{tt}$	E

Problem 4) (10 points)

a) (3 points) A surface S is parameterized by

$$\vec{r}(u,v) = [u, v, uv] \ ,$$

where $u^2 + v^2 \leq 1$. Find its surface area.



b) (3 points) Parametrize the boundary curve C matching the orientation $\vec{r}_u \times \vec{r}_v$ of S, then compute the line integral $\int_C \vec{F} \cdot \ d\vec{r}$ with $\vec{F}(x,y,z) = [-y, x, 1]$.

c) (2 points) The coordinates of the surface S satisfy $xy - z = 0$. Find the tangent plane to S at $(2,1,2)$.

d) (2 points) Find the linearization $L(x,y)$ of $f(x,y) = xy$ at the point $(2,1)$.

Problem 5) (10 points)

On November 17 2017, the **NASA Eagleworks paper** appeared, making the **EM drive** more probable. It might in future be used for deep space missions. The drive produces a thrust, apparently violating momentum conservation.

a) (5 points) Assume the drive flies in the gravitational field

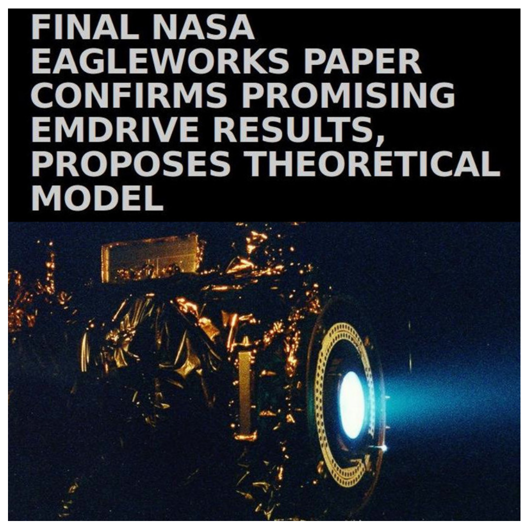
$$\vec{F}(x,y,z) = [x^7 + xy^2z^2, x^2yz^2, x^2y^2z]$$

along the path

$$C : \vec{r}(t) = [t\cos(t), t\sin(t), t(5\pi - t)]$$

with $0 \leq t \leq 5\pi$. Compute the work

$$\int_0^{5\pi} \vec{F} \ \cdot \ d\vec{r}.$$



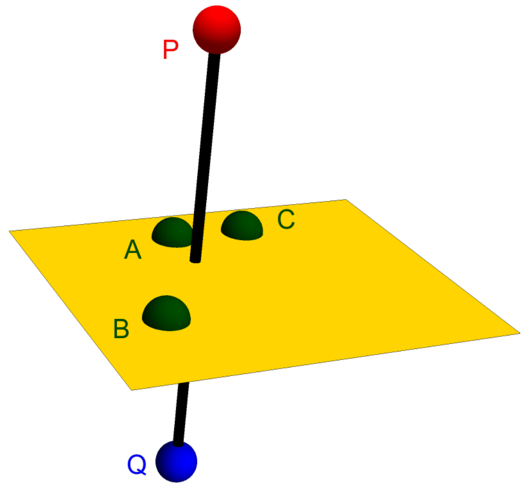
b) (3 points) Compute $d = |\int_0^{5\pi} \vec{r}\,'(t) \ dt|$.

c) (2 points) If L is the arc length of C, circle the one box below which applies:

$d = L$	$d > L$	$d < L$

Problem 6) (10 points)

a) (2 points) Find the equation $ax+by+cz=d$ of the plane through $A=(1,1,1), B=(3,4,5), C=(4,4,2)$.



b) (3 points) Compute the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AC}$.

c) (3 points) Determine the volume of the parallelepiped spanned by $\vec{AB}, \vec{AC}, \vec{AP}$ where $P=(1,3,4)$.

d) (2 points) Find the distance $|\vec{PQ}|$, where Q is the mirror image of P opposite of the plane. It is determined by the fact that the middle point $(P+Q)/2$ is on the plane and that $\vec{PQ}$ is perpendicular to the plane.

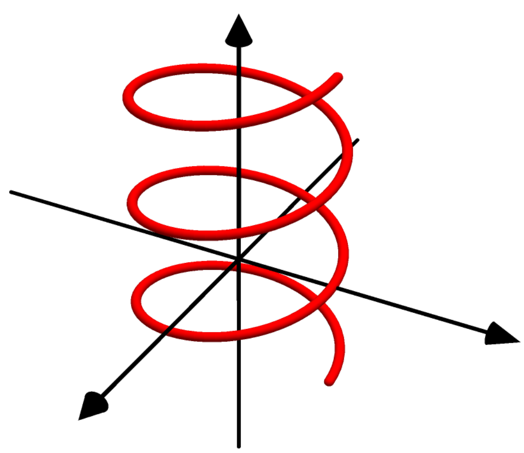
Problem 7) (10 points)

The triple scalar product is also written as

$$[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}).$$

The **torsion** of a space curve is defined as

$$\frac{[\vec{r}\,', \vec{r}\,'', \vec{r}\,''']}{|\vec{r}\,' \times \vec{r}\,''|^2}.$$



a) (3 points) Compute $\vec{r}\,'(0), \vec{r}\,''(0), \vec{r}\,'''(0)$ for

$$\vec{r}(t) = [\cos(t), \sin(t), t].$$

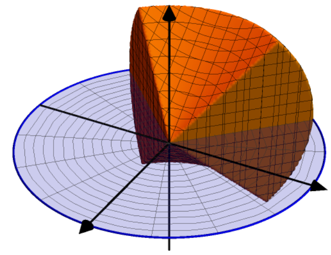
b) (4 points) Compute the torsion of the curve at the point $\vec{r}(0)$.

c) (3 points) Assume you have an arbitrary curve $\vec{r}(t)$ which is contained in the xy-plane. What is its torsion?

Problem 8) (10 points)

Let E be the solid

$$x^2 + y^2 \geq z^2, x^2 + y^2 + z^2 \leq 9, y \geq |x|.$$



a) (7 points) Integrate

$$\iiint_E x^2 + y^2 + z^2 \; dxdydz.$$

b) (3 points) Let $\vec{F}$ be a vector field

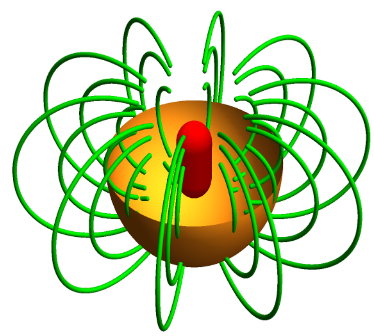
$$\vec{F} = [x^3, y^3, z^3].$$

Find the flux of $\vec{F}$ through the boundary surface of E, oriented outwards.

Problem 9) (10 points)

The vector field

$$\vec{A}(x, y, z) = \frac{[-y, x, 0]}{(x^2 + y^2 + z^2)^{3/2}}$$



is called the **vector potential** of the magnetic field

$$\vec{B} = \text{curl}(\vec{A}).$$

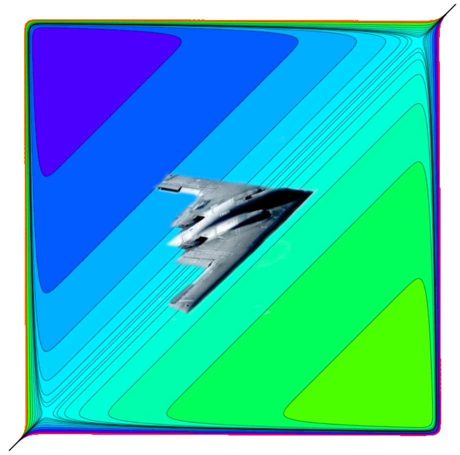
The picture shows some flow lines of this **magnetic dipole field** $\vec{B}$. Find the flux of $\vec{B}$ through the lower half sphere $x^2 + y^2 + z^2 = 1, z \leq 0$ oriented downwards.

Problem 10) (10 points)

a) (8 points) Classify the critical points of the **area 51** function

$$f(x,y) = x^{51} - 51x - y^{51} + 51y$$

using the second derivative test. The reason why this function was chosen is classified.



b) (2 points) Does the function have a global maximum or global minimum on the region $x^2+y^2 \leq 1$ including the boundary? Write "yes" or "no" with a brief explanation. There is no need to find the global extrema.

Problem 11) (10 points)

Using the Lagrange method, find the maximum and minimum of the elliptic curve function

$$f(x,y) = y^2 - x^3 - x^2 - x$$

on the circle $g(x,y) = x^2 + y^2 = 1$.

This problem is motivated from a real life application. To encrypt communication in **"WhatsApp"**, the elliptic curve 25519 given by $y^2 = x^3 + 486662x^2 + x$ over the prime $p = 2^{255} - 19 =$

57896044618658097711785492504343953926634992332820282019728792003956564819949

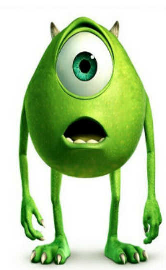
is used.

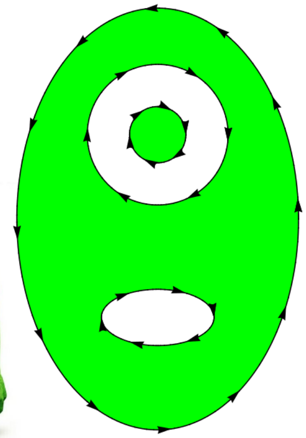
Problem 12) (10 points)

Given the scalar function $f(x, y) = x^5 + xy^4$, compute the line integral of

$$\vec{F}(x, y) = [5y + 3y^2, 6xy + y^4] + \nabla f$$

along the boundary of the **Monster region** given in the picture. There are four boundary curves, oriented as shown in the picture: a large ellipse of area 16, two circles of area 1 and 2 as well as a small ellipse (the mouth) of area 3. The picture describes the orientations of the boundary curves perfectly and they are as they are! We warn you not to ask about this, or else we will bring in "Mike" from **Monsters, Inc.**





Problem 13) (10 points)

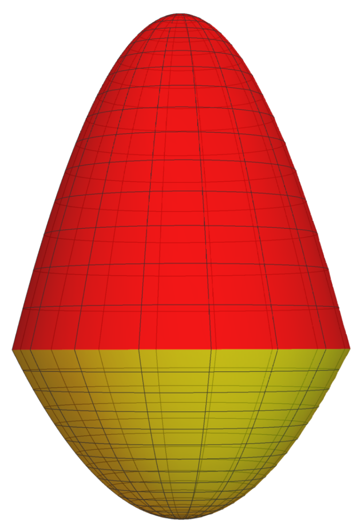
"ProtEgg" is a defense spell. It produces an egg shaped solid E enclosed by the surfaces

$$S : z = 2 - 2x^2 - 2y^2, z \geq 0 ,$$

where S is oriented upwards and

$$T : z = x^2 + y^2 - 1, z \leq 0 ,$$

where T is oriented downwards.



a) (4 points) Find the volume $\iiint_E 1 \, dxdydz$ of E.

b) (4 points) The spell uses a force field

$$\vec{F}(x, y, z) = [0, 0, 2x^2 + 2y^2 + z] .$$

S is parametrized by $\vec{r}(u, v) = [u, v, 2 - 2u^2 - 2v^2]$ with $u^2 + v^2 \leq 1$ oriented upwards. Compute the flux $\iint_S \vec{F} \cdot d\vec{S}$ without an integral theorem.

c) (2 points) The flux $\iint_T \vec{F} \cdot d\vec{S}$ can be determined using an integral theorem. What is the value of the flux? Check all that apply:

A	B	C	D
$\iint_S \vec{F} \cdot d\vec{S}$	$-\iint_S \vec{F} \cdot d\vec{S}$	$\iiint_E 1 \, dV + \iint_S \vec{F} \cdot d\vec{S}$	$\iiint_E 1 \, dV - \iint_S \vec{F} \cdot d\vec{S}$

Problem 14) (10 points)

Archimedes computed the volume of the intersection of three cylinders. The **Archimedes Revenge** is the problem to determine the volume V of the solid R defined by

$$x^2+y^2-z^2 \leq 1, y^2+z^2-x^2 \leq 1, z^2+x^2-y^2 \leq 1 .$$

Archimedes Revenge is brutal! It is definitely to hard for this exam. We give you therefore the volume $V = \log(256)$. Now to the **actual exam problem**: find the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of

$$\vec{F}(x,y,z) = [2x+y^2+z^2, x^2+2y+z^2, x^2+y^2+2z]$$

through the boundary surface S of R, assuming that S is oriented outwards.