Name:

MWF 9 Oliver Knill
MWF 9 Arnav Tripathy
MWF 9 Tina Torkaman
MWF 10:30 Jameel Al-Aidroos
MWF 10:30 Karl Winsor
MWF 10:30 Drew Zemke
MWF 12 Stepan Paul
MWF 12 Hunter Spink
MWF 12 Nathan Yang
MWF 1:30 Fabian Gundlach
MWF 1:30 Flor Orosz-Hunziker
MWF 3 Waqar Ali-Shah

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and nice unless stated otherwise.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

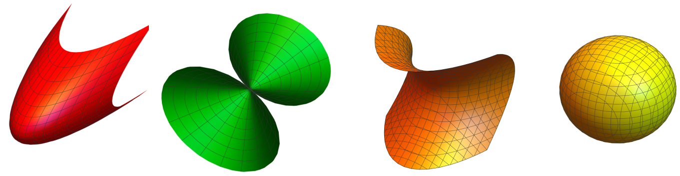
1)	T	F	The parametrization $\vec{r}(u,v) = [v^2, v^2\cos(u), v^2\sin(u)]$ describes a cone.
2)	T	F	If three vectors $\vec{u},\vec{v},\vec{w}$ satisfy $\vec{u}\cdot\vec{v}=0$ and $\vec{v}\times\vec{w}=\vec{0}$, then $\vec{u}\cdot\vec{w}=0$.
3)	T	F	Let S be a surface bounding a solid E and $\vec{F}$ is a vector field in space which is incompressible, $\text{div}(\vec{F})=0$, then $\iint_S \vec{F}\cdot d\vec{S}=0$.
4)	T	F	If $\text{curl}(\vec{F})(x,y,z)=0$ for all (x,y,z) then $\iint_S \vec{F}\cdot d\vec{S}=0$ for any closed surface S.
5)	T	F	If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ has zero curl everywhere.
6)	T	F	If $\vec{F},\vec{G}$ are two vector fields which have the same divergence then $\vec{F}-\vec{G}$ is constant.
7)	T	F	The linearization of the constant function $f(x,y)=3$ at $(x,y)=(1,1)$ is the function $L(x,y)=0$.
8)	T	F	The surface area of a surface S is $\int\int_S [x,y,z]\cdot d\vec{S}$.
9)	T	F	There is a non-constant vector field $\vec{F}(x,y,z)$ such that $\text{curl}(\vec{F}) = \text{curl}(\text{curl}(\vec{F}))$
10)	T	F	If $\vec{F}$ is a vector field and E is the unit ball then $\iiint_E \text{div}(\text{curl}(\vec{F}))\,dV=0$.
11)	T	F	If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S is zero.
12)	T	F	The equation $\text{grad}(\text{div}(\text{grad}(f)))=\vec{0}$ always holds.
13)	T	F	The vector $\vec{k}\times(\vec{j}\times\vec{i})$ is the zero vector, if $\vec{i}=[1,0,0], \vec{j}=[0,1,0]$, and $\vec{k}=[0,0,1]$.
14)	T	F	If f is minimized at (a,b) under the constraint $g=c$, then $\nabla f(a,b)$ and $\nabla g(a,b)$ are perpendicular.
15)	T	F	If A,B,C,D are four points in space such that the line through A,B intersects the line through C,D, then A,B,C,D lie on some plane.
16)	T	F	The chain rule assures that for a vector field $\vec{F}=[P,Q,R]$ the formula $\frac{\partial}{\partial u}\vec{F}(\vec{r}(u,v)) = [\nabla P\cdot\vec{r}_u, \nabla Q\cdot\vec{r}_u, \nabla R\cdot\vec{r}_u]$ holds.
17)	T	F	The vector field $\vec{F}=[\cos(y),0,\sin(y)]$ satisfies $\text{curl}(\vec{F})=\vec{F}$. By the way, it is called the Cheng-Chiang field.
18)	T	F	The unit tangent vector $\vec{T}(t)$, the normal vector $\vec{N}(t)$ and the binormal vector $\vec{B}(t)$ for a given curve $\vec{r}(t)$ span a cube of volume 1 at $t=1$.
19)	T	F	The vector field $\vec{F}(x,y,z)=[z,z,z]$ can not be the curl of a vector field.
20)	T	F	The expression $\text{curl}(\text{grad}(\text{div}(\text{grad}(\text{div}(\text{curl}(\vec{F}))))))$ is a well defined vector field in three dimensional space.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$\vec{r}(u,v) = [u, u^2 - v^2, v]$	
$y - x^2 - z^2 = 0$	
$\vec{r}(u,v) = [u\cos(v), u, u\sin(v)]$	
$x^2 + y^2 = 1 - z^2$	

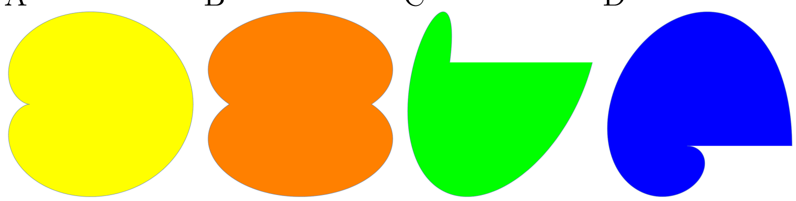
1 2 3 4



b) (2 points) Match the following regions given in polar coordinates (r, θ):

Region	A-D
$r < \theta^2$	
$r < 1 + \cos(\theta)$	
$r < 1 + \lvert\sin(\theta)\rvert$	
$r < (4\pi^2 - \theta^2)$	

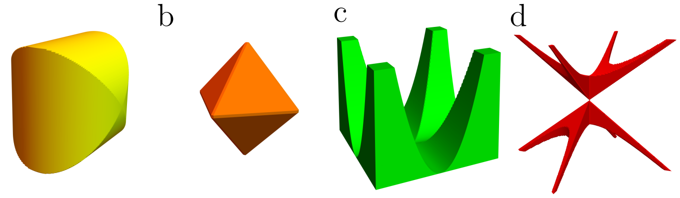
A B C D



c) (2 points) Match the regions. There is an exact match.

Solid	a-d
$\lvert x\rvert + \lvert y\rvert + \lvert z\rvert \leq 1$	
$x^2 + y^2 \leq 1, x^2 + z^2 \leq 1$	
$x \leq y^2, x \leq z^2$	
$0 \leq x^2 - y^2 \leq 4, 0 \leq y^2 - z^2 \leq 4$	

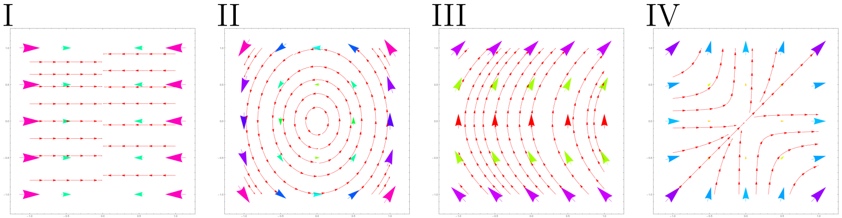
a b c d



d) (2 points) The figures display vector fields in the plane. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = [y, 1]$	
$\vec{F}(x,y) = [-2y, 3x]$	
$\vec{F}(x,y) = [-x, 0]$	
$\vec{F}(x,y) = [x^2, y^2]$	

I II III IV



e) (1 point) Find the linearization $L(x,y)$ of $f(x,y) = xy$ at $(2,1)$.

f) (1 point) Write down the wave equation for a function $f(t,x)$:

Problem 3) (10 points)

a) (6 points) In the following, f is a function, $\vec{F}$ a vector field, I is an interval, G is a region in the two dimensional plane, S is a closed surface parametrized by $\vec{r}(u,v)$, C is a closed curve parametrized by $\vec{r}(t)$ and E is a solid. Fill the blanks using "volume", "area", "length" or "0", where a choice can appear a multiple times.

$$\boxed{} = \int\int_G \text{curl}([x,0])\ dxdy$$

$$\boxed{} = \int\int_G \text{curl}([-y,0])\ dxdy$$

$$\boxed{} = \int\int\int_E \text{div}([x,x,x])\ dxdydz$$

$$\boxed{} = \int\int_S |\vec{r}_u \times \vec{r}_v|\ dudv$$

$$\boxed{} = \int_C |\vec{r}\,'(t)|\ dt + \int_C \text{grad}(f)\ \cdot d\vec{r}$$

$$\boxed{} = \int\int_S \text{curl}(\vec{F})\ \cdot d\vec{S}$$

b) (4 points)
Complete the formulas of the following boxes

$\int_0^{2\pi}\int_0^{\pi}\int_0^1 \boxed{}\ d\rho d\phi d\theta$ Integral for volume of unit ball $x^2+y^2+z^2 \leq 1$.

$\int_0^{2\pi}\int_0^1 \boxed{}\ drd\theta$ Integral for area of unit disc $x^2+y^2 \leq 1$

$g_x(x,y) = \dfrac{\boxed{}}{f_z(x,y,z)}$ Implicit derivative of g satisfying $f(x,y,g(x,y)) = 0$

$|\vec{u} \cdot (\boxed{})|$ Volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

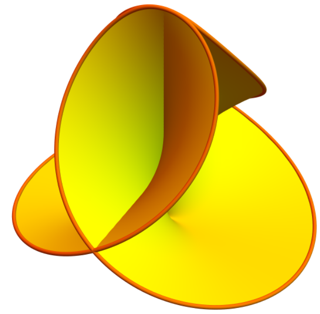
Problem 4) (10 points)

Find the surface area of the surface parametrized by

$$\vec{r}(u,v) = [u\cos(v), u\sin(v), \frac{v^2}{2}] \ ,$$

where (u,v) are in the domain $u^2+v^2 \leq 9$.

P.S. You do not have to worry that this cool surface has self-intersections.



Problem 5) (10 points)

On planet **Tatooine**, Luke Skywalker travels along a path C parametrized by

$$\vec{r}(t) = [t\cos(t), t\sin(t), 0]$$

from $t=0$ to $t=2\pi$. What is the work done

$$\int_C \vec{F} \cdot \vec{dr}$$

by the "force"

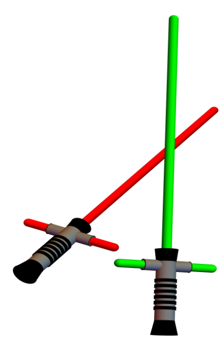
$$\vec{F} = [x^2+y+z, y^3+x, z^5+x] \ .$$

Problem 6) (10 points)

Tomorrow, on December 18, the "force awakens". There will be light sabre battles, without doubt.

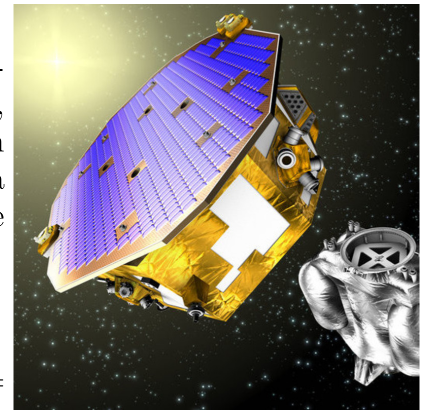
a) (7 points) What is the distance between two light sabres given by cylinders of radius 1 around the line $\vec{r}(t) = [t, -t, t]$ and the line connecting $(0,14,0)$ with $(3,5,6)$.

b) (3 points) A spark connects the two points of the sabre which are closest to each other. Find a vector in that direction.



Sabre by Connor (mathematica project).

Problem 7) (10 points)



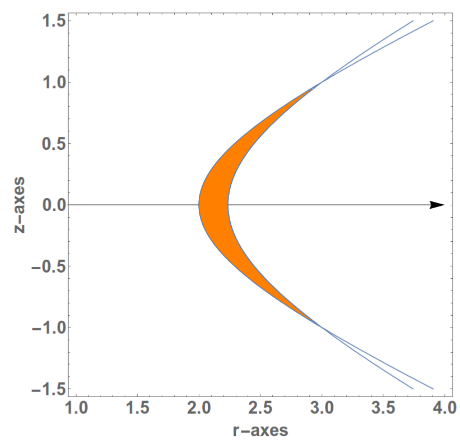
LISA proof of concept will be followed by eLISA in 2034.

100 years ago, Einstein proposed gravitational waves. To measure them, the LISA pathfinder was launched on December 3, 2015. It carries two cubes to the Lagrangian point between Earth and Moon aiming to measure the waves. Assume a gravitational wave from a black hole merger produces a force leading to an acceleration

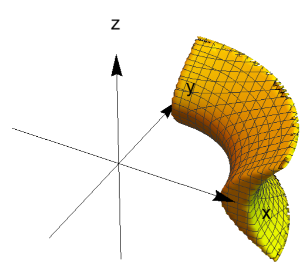
$$\vec{r}\,''(t) = [\sin(t), \cos(t), \sin(t)] \ .$$

What is $\vec{r}(t)$ at time $t = \pi$ if $\vec{r}\,'(0) = [2, 0, 0]$ and $\vec{r}(0) = [1, 0, 3]$.

Problem 8) (10 points)



a) (5 points) Find the integral $\int\int_G r \ drdz$, where G is the region enclosed by the curves $r^2 - 4z^2 = 5$ and $r^2 - 5z^2 = 4$ and contained in $r \geq 0$.



b) (5 points) The Galactic Empire builds a space craft E given as a solid in $x \geq 0, y \geq 0$, enclosed by

$$x^2 + y^2 - 4z^2 = 5$$

and

$$x^2 + y^2 - 5z^2 = 4 \ .$$

Find its volume. P.S. You can make use of problem a) to solve part b) as the problems are related.

Problem 9) (10 points)

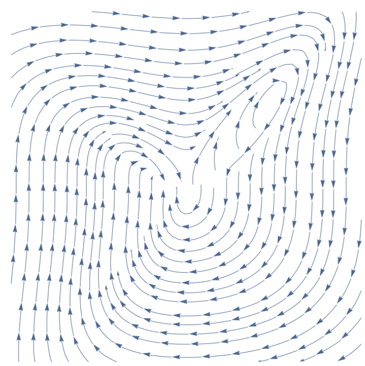
In September 2015, the west side of the Harvard Science center honored the concept of curl by displaying paddle wheels. One of the wheel tips moves on an oriented curve $\vec{r}(t) = [\cos(t), 0, \sin(t)]$ bounding the disc parametrized by $\vec{r}(u,v) = [u\cos(v), 0, u\sin(v)]$ with $0 \leq u \leq 1, 0 \leq v \leq 2\pi$. Let $\vec{F}$ be the wind vector field

$$\vec{F}(x,y,z) = [x^9 + y^7 + 3z, x^9 + y^9 + \sin(z), z^5 e^z] .$$

Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ measuring the energy gain during one rotation along the curve C parametrized by $\vec{r}(t)$.

Problem 10) (10 points)

The value of the line integral of the vector field $\vec{F}(u,v) = (2/\pi)[-uv^2 + v^3, uv - u^3]$ along a curve $\vec{r}(t) = [x + \cos(t), y + \sin(t)]$ depends only on the center point (x,y) and is given by



$$f(x,y) = -3 - 6x^2 + 2y + 4xy - 6y^2 .$$

a) (7 points) Find all critical points (x,y) for the function $f(x,y)$ and analyze them using the second derivative test.
b) (3 points) Given that

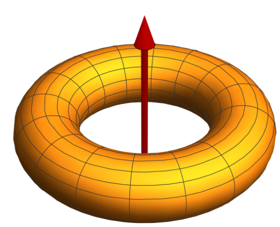
$$f(x,y) = -3 - (x-2y)^2 - 5x^2 - 2y^2 + 2y ,$$

decide whether there is a global maximum for f.

Problem 11) (10 points)

The moment of inertia $f(x,y)$ of a torus of mass 4 with smaller tube radius x and bigger center curve radius y is

$$f(x,y) = 3x^2 + 4y^2 .$$



a) (7 points) Find the parameters (x_0, y_0) for the torus which have minimal moment of inertia under the constraint that

$$g(x,y) = x + 4y = 13 .$$

b) (3 points) Write down the equation of the tangent line to the level curve of f which passes through (x_0, y_0).

Problem 12) (10 points)

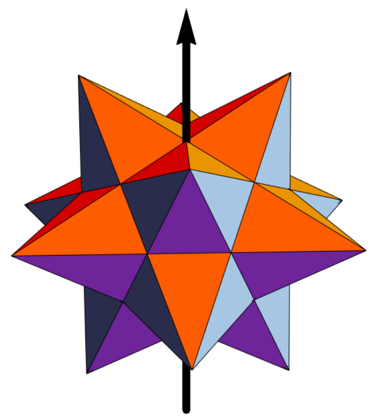
A new **elliptical machine** has been designed to simulate running better. The leg of a runner moves on the curve parametrized by

$$\vec{r}(t) = [8\cos(t), 2\sin(t) + \sin(2t) + \cos(2t)]$$

with $0 \leq t \leq 2\pi$. Find the area of the region enclosed by the curve.

Problem 13) (10 points)

The polyhedron E in the figure is called **small stellated Dodecahedron**. The solid E has volume 10. Its moment of inertia $\iiint_E x^2 + y^2 \; dxdydz$ around the z-axis is known to be 1. Let S be the boundary surface of the polyhedron solid E oriented outwards.



a) (5 points) What is the flux of the vector field

$$\vec{F}(x, y, z) = [y^5 + x, z^5 + y, x^5 + z]$$

through S?

b) (5 points) What is the flux of the vector field

$$\vec{G}(x, y, z) = [x^3/3, y^3/3, 0]$$

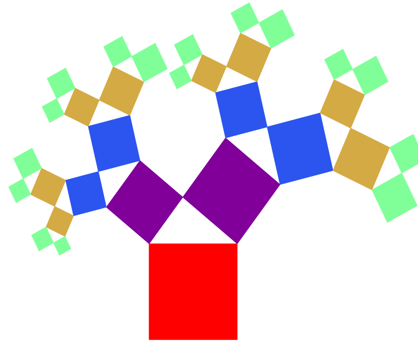
through S?

Problem 14) (10 points)

Find the line integral of the vector field

$$\vec{F}(x, y) = [-y + x^8, x - y^9]$$

along the boundary C of the generation 4 **Pythagoras tree** shown in the picture. The curve C traces each of the 31 square boundaries counter clockwise. You can use the Pythagoras tree theorem mentioned below. We also included the proof of that theorem evenso you do not need to read the proof in order to solve the problem.



Pythagoras tree theorem:

The generation n Pythagorean tree has area $n + 1$.

Proof: in each generation, new squares are added along a right angle triangle. The 0'th generation is a square of area $c^2 = 1$. The first generation tree got two new squares of side length a, b which by **Pythagoras** together have area $a^2 + b^2 = c^2 = 1$. Now repeat the construction. In generation 2, we have added 4 new squares which together have area 1 so that the tree now has area 3. In generation 3, we have added 8 squares of total area 1 so that the generation tree has area 4. Etc. Etc. The picture to the right shows generation 7. Its area of all its (partly overlapping) leafs is 8.