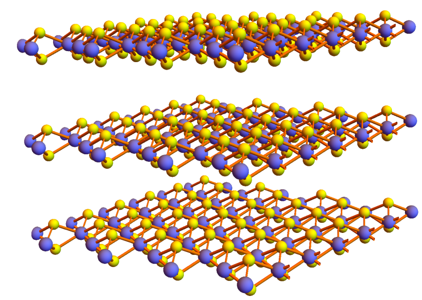
Review Final Math 21a, Dec/9/2019

1) Geometry

Molybdenum disulfide MoS_2 consists of molybdenum Mo and sulfur S. Crystalized molybdenite can be found for example in the Swiss mountains, like in the Baltschieder region. The crystal layer structure is pretty cool. Similarly as graphite, the planes are held together by van der Waals forces. As it has low friction and is added as a solid lubricant. Assume one of the layers is the plane

$$x + 2y + z = 4$$

and on the next layer is an atom with coordinate $P = (3, 4, 5)$.



a) (7 points) What is the distance of P to the plane?

b) (3 points) Please parametrize a line perpendicular to the plane passing through P.

Solution:
a) We have $\vec{n} = [1, 2, 1]$. Take a point on the plane link $Q = (4, 0, 0)$. Now the distance is $|\vec{PQ} \cdot n|/\sqrt{n} = |[1, -4, 5] \cdot [1, 2, 1]|/\sqrt{6} = 12/\sqrt{6}$.
b) We have

$$\vec{r}(t) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} .$$

2) Gradients

The book **"Magnificant Mistakes in Mathematics"** reports that Legendre conjectured in a book of 1794 that there are no rational solutions of the equation

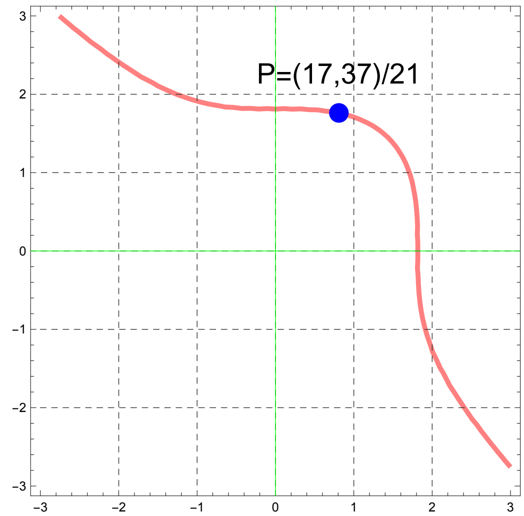
$$x^3 + y^3 = 6 \ .$$

Henry Ernest Dudeny proved this wrong by stating $(17/21)^3 + (37/21)^3 = 6$.

a) (3 points) Find a nonzero vector $[a, b]$ perpendicular to the curve at $(17/21, 37/21)$.

b) (4 points) Find the tangent line to the curve at $(17/21, 37/21)$.

c) (3 points) Find the linearization $L(x, y)$ at that point.



Solution:

a) The gradient is $\nabla f(x, y) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}$. Evaluated at the point gives $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3(17/21)^2 \\ 3(37/21)^2 \end{bmatrix}$.

b) The equation of the line is $ax + by = d$, where a, b were already computed in a) and where d is obtained by plugging in the point. The answer is

$$3(17/21)^2 x + 3(37/21)^2 y = 18$$

The number on the right came from the fact that $x_0^3 + y_0^3 = 6$.

c) The linearization is

$$L(x, y) = 6 + a(x - 17/21) + b(y - 37/21) = 6 + 3(17/21)^2(x - 17/21) + 3(37/21)^2(y - 37/21) \ .$$

3) Extrema

In order to optimize a parachute designed by Da Vinci, we classify the critical points of the function

$$f(x, y) = 4x^2y + 2x^2 + y^2 \ .$$

The first part is related to the volume, the second part is related to residual volume dragged along during the fall. Don't worry about the derivation of the function $f(x, y)$. It is a Da Vinci thing. So:

a) (8 points) Classify the critical points.

b) (2 points) Decide whether any of the points is a global maximum or global minimum.

Solution:
a) This is a standard extremization problem. We want to find the points where the gradient $\nabla f(x, y) = [8xy + 4x, 4x^2 + 2y]$ is zero. For the first equation there are the possibilities $x = 0$ or $y = -1/2$. From the second equation we get so the x values $x = \pm 1/4 = \pm 1/2$. The three critical points are $(0, 0)$, $(1/2, -1/2)$ and $(-1/2, 1/2)$. We compute $f_{xx} = 8y_2$ and $f_{yy} = 2$ and $f_{xy} = 8x$. Then compute $D = f_{xx}f_{yy} - f_{xy}^2$ at each point

Point	D	f_{xx}	Nature	Value
$(-\frac{1}{2}, -\frac{1}{2})$	-16	0	saddle	$\frac{1}{4}$
$(0, 0)$	8	4	minimum	0
$(\frac{1}{2}, -\frac{1}{2})$	-16	0	saddle	$\frac{1}{4}$

b) If we put $y = x$, we have the function $4x^3 + 3x^2$ which grows indefinitely for $x \to \infty$, both to infinity ($x \to \infty$) and to minus infinity ($x \to -\infty$).

4) Constraints

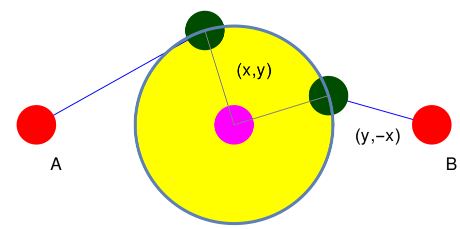
Motivated by a drawing of Da Vinci, we look at the following problem: a wheel of radius 1 is attached by rubber bands to two points $A = (-2, 0)$ and $B = (2, 0)$. The point (x, y) connects to A and $(y, -x)$ to $(2, 0)$. The point (x, y) is constrained to

$$g(x, y) = x^2 + y^2 = 1 \; .$$

The wheel will settle at the position, for which the potential energy $f(x, y) = [(x + 2)^2 + y^2 + (y - 2)^2 + x^2]/2$ which is

$$f(x, y) = x^2 + y^2 + 2x - 2y + 4$$

is minimal. Find that position using the method of Lagrange multipliers.

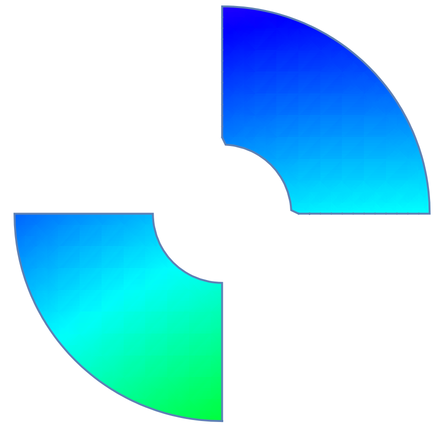


Solution:
The **Lagrange equations** are

$$\begin{aligned} 2x + 2 &= \lambda(2x) \\ 2y - 2 &= \lambda(2y) \\ x^2 + y^2 &= 1 \end{aligned}$$

Eliminating λ gives $y = -x$. Plugging into the third equation gives $x^2 + (-x)^2 = 1$ so that $(x, y) = (-1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (1/\sqrt{2}, -1/\sqrt{2})$. Evaluating the function f on the two points shows that $(-1/\sqrt{2}, 1/\sqrt{2})$ is the minimum.

5) Integration



a) (5 points) Evaluate the triple integral

$$\int_0^4 \int_0^{x^2} \int_0^1 \frac{y^4}{4-\sqrt{y}} \, dzdydx \; .$$

b) (5 points) Find the moment of inertia

$$\iiint_E x^2 + y^2 \; dzdxdy$$

for the region $E = \{xy > 0, \;\; 4 < x^2 + y^2 < 9 \; 0 \leq z \leq 1\}$.

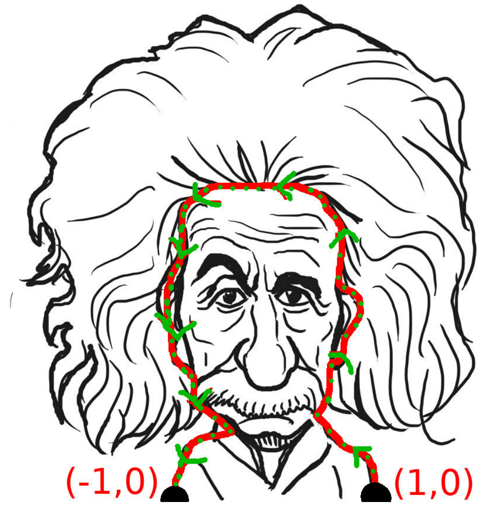
Solution:
a) This is a case for a change of integration order. Make a picture. We get

$$\int_0^{\infty} \int_{\sqrt{y}}^4 \frac{y^4}{4-\sqrt{y}} \; dxdy = \int_0^{16} y^4 \; dy = 16^5/5 \; .$$

b) This is a case for polar coordinates. By symmetry, we can integrate over the first quadrant only and multiply by 2. The result is (don't forget the integration factor r, which was also here the most common mistake):

$$2 \int_0^{\pi/2} r^2 r \; dr d\theta \;\; = \frac{65\pi}{4} \; .$$

6) Line integrals



What is the line integral

$$\int_0^1 \vec{F}(r(t)) \cdot \vec{r}\,'(t)\; dt$$

along the **Einstein curve** shown in the picture? The curve goes from $\vec{r}(0) = (1,0)$ to $\vec{r}(1) = (-1,0)$. The vector field is

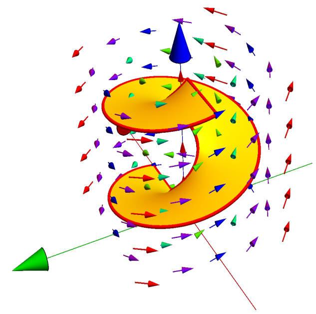
$$\vec{F}(x,y) = \begin{bmatrix} 4x^3 + y + y^2 \\ 1 + x + 2xy \end{bmatrix} .$$

Don't ask for the formula of the Einstein curve. Only Einstein knows.

Solution:
The vector field is a gradient field with potential $f(x,y) = x^4 + xy + y^2x + y$. By the **fundamental theorem of line integrals**, we have

$$\int_0^{2\pi} \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\; dt = f(-1,0) - f(1,0) = 1 - 1 = 0$$

7) Flux integrals



We build a model of the **Da Vinci helicopter**. The helicopter blade is a helix S parametrized as

$$\vec{r}(u,v) = \begin{bmatrix} u\cos(v) \\ u\sin(v) \\ v \end{bmatrix}$$

with $0 \le v \le 3\pi$ and $1 \le u \le 5$. Its boundary C consists of 4 parts. A path going radially out, then the helix up, going radially in and then along the axes down. Let $\vec{F}$ be the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix} .$$

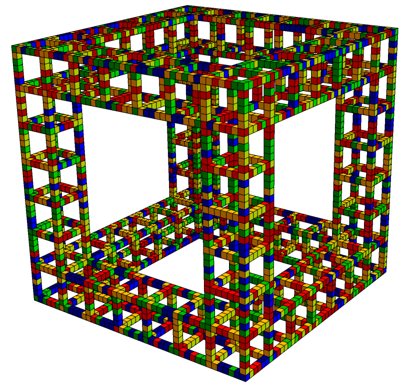
Find the line integral of $\vec{F}$ along the curve C. The curve is oriented so that it is compatible with the surface orientation, which is the orientation given by the parametriation.

Solution:
This is a problem for Stokes theorem. We compute the flux of the field $\text{curl}(\vec{F})$ through the surface. As usual, the most frequently done mistake done here was to compute the flux of $\vec{F}$ and not the flux of the curl. The curl of $\vec{F}$ is $[0,0,2]$. Now compute $\vec{r}_u \times \vec{r}_v = [-\sin(v), -\cos(v), u]$. So, $\text{curl}(\vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 2u$. We have now to integrate this over the domain

$$\int_0^{3\pi} \int_1^5 2u \; dudv = 72\pi \; .$$

8) Integral theorems

We love the **Menger sponge**. For an exhibit hall, we build a variant, where a cube is divided into 7 parts and the middle 5×5 cylinders are cut out, we end up with a fractal of dimension $\log(68)/\log(7) = 2.1684...$. The second iteration E shown in the picture consists of $68^2 = 4624$ cubes of side length 1. What is the flux of the vector field



$$\vec{F}(x, y, z) = \begin{bmatrix} 4x + e^{\cos(y)} \\ z^5 - y \\ y^5 - z \end{bmatrix}$$

through the boundary S of the solid E assuming as usual that the surface S is oriented outwards?

Solution:
The divergence of $\vec{F}$ is constant and equal to 2. By the **divergence theorem**, the result is 2 times the volume of the solid E. Which is $2 * 4624 = 9248$.

9) Area

The **"Easy-going region"** D enclosed by the curve

$$\vec{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) - \cos(t) \end{bmatrix}$$

with $0 \le t \le 2\pi$ is called the "laidback disk". You can just call it the "Dude" or "His Dudeness" if you are not into that whole brevity thing. What is the area of the dude D?

Solution:
This is a problem for **Green's theorem**. Take the vector field $\vec{F} = [0, x]$ as usual. Then compute the line integral of $\vec{F}$ along the boundary.

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_0^{2\pi} \begin{bmatrix} 0 \\ \cos(t) - \cos(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) + \sin(t) \end{bmatrix} dt$$

This is $\int_0^{2\pi} \cos^2(t) + \sin(t)\cos(t)\; dt = \int_0^{2\pi} (1 + \cos(2t))/2 + \sin(2t)/2\; dt = \pi$.

10) True false

True False

1) T **F** The angle between the vectors $[1, 2, 3]$ and $[2, -1, -1]$ is $\frac{\pi}{6}$.

Solution:
The dot product is negative. It is an obtuse angle and can not be $30°$.

2) **T** F The function $u(x, y) = x^2 + y^2$ is a solution of the wave equation.

Solution:
Yes, $u_{xx} = 2$ and $u_{yy} = 2$.

3) **T** F $[4, 6, 8]$ is a normal vector for the plane $-2x - 3y - 4z = 5$.

Solution:
Yes, it is parallel to the gradient vector $[-2, -3, -4]$ of the plane.

4) **T** F The directional derivative of a function f in the direction of $\nabla f/|\nabla f|$ can never be negative.

Solution:
$D_{\nabla f/|\nabla f|}f = \nabla f \cdot \nabla f/|\nabla f| = |\nabla f| \geq 0$.

5) **T** F The surface parameterized by $[\sin u, \cos v, u^2 + v^2]$, $0 \leq u, v \leq 1$ has the same surface area as the surface parameterized by $[\sin u^2, \cos v^2, u^4 + v^4]$, $0 \leq u, v \leq 1$.

Solution:
Yes, surface area does not depend on the parametrization.

6) **T** F By the chain rule, $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$.

Solution:
Yes, this is the proof of the fundamental theorem of line integrals.

7) | T | F | Let C be the unit circle parametrized counter-clockwise. If $\vec{F}(x,y)$ is a vector field and $\int_C \vec{F} \cdot d\vec{r} = 0$, then $\vec{F}$ is a gradient vector field.

Solution:
It is not sufficient to check the closed loop property for one closed path only. It has to be true for all closed paths.

8) | T | F | The vector field $\vec{F}(x,y,z) = [y^2 - z^2, z^2 - x^2, x^2 - y^2]$ is conservative.

Solution:
It is incompressible because the divergence is zero but it is not conservative, because the curl is not identically zero. Already the third component is not.

9) | T | F | The vector field $\vec{F}(x,y,z) = [x,y,z]$ is the curl of a vector field.

Solution:
If it were of the form $\vec{G} = \text{curl}(\vec{F})$ then the divergence would have to be zero.

10) | T | F | For any vector field $\vec{F} = [P,Q,R]$, the identity $\text{grad}(\text{div}(\text{curl}(\vec{F})) = \text{div}(\text{grad}(\text{div}(\vec{F})))$ holds.

Solution:
They are different in general. Take $\vec{F} = [0,0,z^3]$ for example. The left hand side is zero, the right hand side is 6.

11) | T | F | The equation $\text{div}(\text{grad}(\text{div}(\text{grad}(f))) = 0$ is always true for any smooth function f.

Solution:
take $f(x,y,z) = x^4$ for example.

12) T **F** A surface S has boundary C with compatible orientation then $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt = \iint_S \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v\ dudv$.

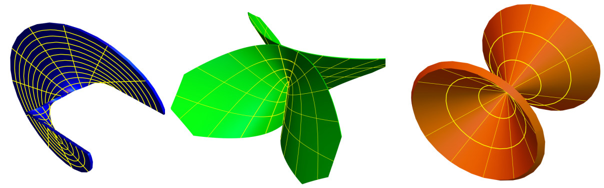
Solution:
We would have to take the curl of F.

11 Matching

a) (2 points) Match the following surfaces. There is an exact match.

Parametrized surface $\vec{r}(s,t)$	A-C
$[v\cos(u), v\sin(u), u]$	
$[u\cos(v), u, u\sin(v)]$	
$[u^2v, uv^2, u^2 - v^2]$	

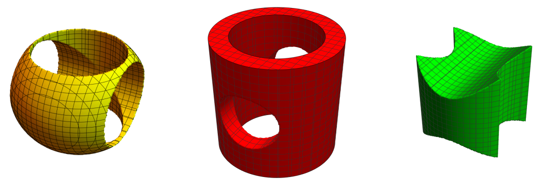
A B C



b) (2 points) Match the solids. There is an exact match.

Solid	A-C
$x^2 + y^2 + z^2 < 9, x^2 + y^2 > 4, y^2 + z^2 > 4$	
$x^2 - y^2 < 1, z^2 - x^2 < 1, x^2 + y^2 < 4$	
$x^2 + y^2 < 4, x^2 + y^2 > 2, x^2 + z^2 > 1$	

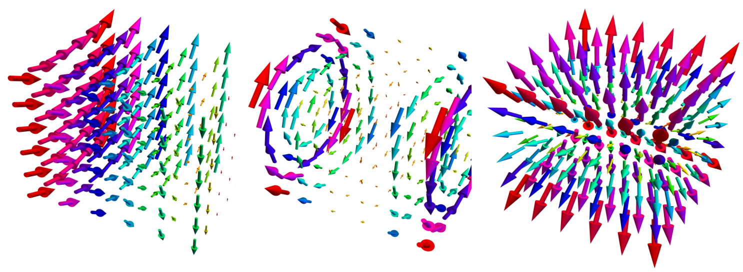
A B C



c) (2 points) The figures display vector fields. There is an exact match.

Field	A-C
$\vec{F} = [0, -x\sin(z), x\sin(y)]$	
$\vec{F} = [0, 1-x, x]$	
$\vec{F} = [x, 2y, 3z]$	

A B C



d) (2 points) Recognize partial differential equations!

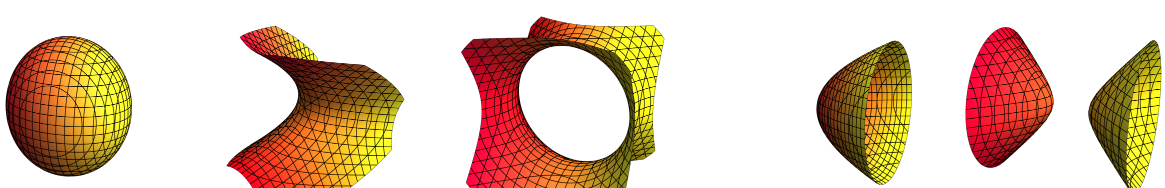
Equation	A-F
Laplace	
Burger	
Wave	

	PDE
A	$X_{tt} = -X_{xx}$
B	$X_t^2 - 1 = X_x^2$
C	$X_t = X_x$

	PDE
D	$X_t = X_x X_{xx}$
E	$X_{tt} = X_{xx}$
F	$-XX_x = X_t$

e) (2 points) Some surfaces

	Enter a letter from A-E each
Pick the one sheeted hyperboloid	
Pick the one elliptic paraboloid	



A B C D E

Solution:
a) ACB
b) ACB
c) BAC
d) AEF
e) CD

12 Theorems!

We list here **12 of the most important theorems in multivariable calculus** and identify what they are about, then vote which is our favorate. The statement "involves vector fields" means that the statement of the theorem involves a vector field. The statement "involves integrals" means that when writing down the theorem, there appears at least one integral.

Number	Theorem	Involves vector fields	Involves integrals
1)	Pythagoras theorem		
2)	Al Khashi theorem		
3)	Cauchy-Schwarz theorem		
4)	Clairaut theorem		
5)	Fubini theorem		
6)	Gradient theorem		
7)	Second derivative test		
8)	Lagrange theorem		
9)	Fundamental theorem of line integrals		
10)	Greens theorem		
11)	Stokes theorem		
12)	Divergence theorem		

Solution:
The theorems which involve vector fields are the four theorems on vector fields The "Fundamental theorem of line integrals", Green, Stokes and Divergence theorems. The other theorms might involve vectors but not vector fields. The same applies for integrals. These are integral theorems. There was an other theorem, Fubini's theorem, which also involves integrals.

13 Various

If the blank box is replaced by $\nabla f(5,6)$ the statement becomes true or false. Determine which case we have. The function $f(x,y)$ is an arbitrary nice function like for example $f(x,y) = x - yx + y^2$. The curve $\vec{r}(t)$, wherever it appears, parametrizes the level curve $f(x,y) = f(5,6)$ and has the property that $\vec{r}(0) = [5,6]$.

True/False	Topic	Statement
	Linearization	$L(x,y) = f(5,6) + \square \cdot [x-5, y-6]$
	Chain rule	$\frac{d}{dt} f(\vec{r}(t))\|_{t=0} = \square \cdot \vec{r}\,'(0)$
	Steepest descent	f decreases at $(5,6)$ most in the direction of $\square$
	Estimation	$f(5+0.1, 5.99) \sim f(5,6) + \square \cdot [0.1, -0.01]$
	Directional derivative	$D_{\vec{v}} f(5,6) = \square \cdot \vec{v}$, $\|\vec{v}\| = 1$
	Level curve	of f through $(5,6)$ has the form $\square \cdot [x-5, y-6] = 0$
	Vector projection	of $\nabla f(5,6)$ onto $\vec{v}$ is $\vec{v}(\vec{v} \cdot \square)/\|\vec{v}\|^2$
	Tangent line	of $\vec{r}(t)$ at $(5,6)$ is parametrized by $\vec{R}(s) = [5,6] + s\,\square$

Solution:
T,T,F,T,T,F,T,F