12/13/2019 FINAL EXAM **Math 21a, Fall 2019**

Name:

MWF 9 Oliver Knill
MWF 9 Arnav Tripathy
MWF 9 Tina Torkaman
MWF 10:30 Jameel Al-Aidroos
MWF 10:30 Karl Winsor
MWF 10:30 Drew Zemke
MWF 12 Stepan Paul
MWF 12 Hunter Spink
MWF 12 Nathan Yang
MWF 1:30 Fabian Gundlach
MWF 1:30 Flor Orosz-Hunziker
MWF 3 Waqar Ali-Shah

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and defined everywhere unless stated otherwise.
- **Show your work**. Except for problems 1-3 and 6, we need to see details of your computation. If you are using a theorem for example, state the theorem.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) True/False questions (20 points). No justifications are needed.

1) [✓ T] F The distance between a point P and a line L is zero if and only if the point is on the line.

Solution:
It is intuitively clear but also follows from the distance formula as $PQ \times v$ is the zero vector then if Q is any point on the line.

2) T [✓ F] The curves $\vec{r}(t) = [2t, 3t, 0]$ and $\vec{s}(t) = [3t, -2t, 1]$ intersect at a right angle at $(0, 0, 0)$.

Solution:
The velocity vectors are perpendicular but the two curves do not intersect.

3) [✓ T] F The surface $z^2 - (x-3)^2 - (y+1)^2 = -1$ is a one-sheeted hyperboloid.

Solution:
One can see that by putting $z = 0$.

4) T [✓ F] The area of a surface parametrized by $\vec{r}(u, v) = [u, g(u, v), v]$ is given by $\iint_R |g_u \times g_v| \; dudv$.

Solution:
It should be $\vec{r}_u \times \vec{r}_v$ and not $g_u \times g_v$.

5) T [✓ F] If $\vec{r}(0) = \vec{r}\,'(0) = \vec{0}$, then $\vec{r}(t)$ is zero for all times $t > 0$.

Solution:
No, we can have an acceleration .

6) [✓ T] F If $\vec{F} = \nabla f$ and $\vec{r}(t)$ is a curve, then $\frac{d}{dt} f(\vec{r}(t)) = \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$.

Solution:
This is the chain rule.

7) [✓ T] F If $\vec{F}$ and $\vec{G}$ are vector fields in $\mathbf{R}^2$ for which the curl is constant 1 everywhere. Then $\vec{F} - \vec{G}$ is a gradient field.

Solution:
The curl of $\vec{F} - \vec{G}$ is zero.

8) [✓ T] F The solid $1 \leq x^2 + y^2 + z^2 \leq 2$ in $\mathbf{R}^3$ is simply connected.

Solution:
By definition, not every closed curve in this solid can be pulled together to a point.

9) [✓ T] F The parametrization $\vec{r}(u, v) = [u^3, u^6 - v^6, v^3]$ describes a hyperbolic paraboloid (saddle surface).

Solution:
Indeed $x^2 + z^2 = y$.

10) T [✓ F] The flux of the curl of the field $\vec{F}(x, y, z) = [x, y, z]$ through $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ oriented outwards is 4π.

Solution:
By the divergence theorem the flux is zero.

11) T [✓ F] There exists a vector field $\vec{F}$ in space $\mathbf{R}^3$ such that $\text{curl}(\vec{F}) = [5x, -11y, 7z]$.

Solution:
The divergence would have to be zero.

12) T [✓ F] The flux of $\vec{F} = [x, 0, 0]$ through the outwards oriented surface bounding the cuboid $|x| \leq 1, |y| \leq 2, |z| \leq 3$ is equal to 24.

Solution:
It is the volume by the divergence theorem.

13) T [✓ F] The points that satisfy $\theta = \pi/4$ and $\phi = \pi/4$ in spherical coordinates form a surface which is part of a cone.

Solution:
This is a curve, not a surface

14) T [✓ F] If $f(x,y,z)$ is a function and $\vec{F} = \nabla f$ then $\text{div}(\vec{F}) = 0$ everywhere.

Solution:
No, the divergence of the gradient is the Laplacian and not necessarily zero.

15) T [✓ F] For any vector field $\vec{F}$ and any parametrized curve $\vec{r}(t)$ with $t \in [a,b]$ we have $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$.

Solution:
If $\nabla f = \vec{F}$ and we would replace $\vec{F}$ on the right hand side with f, we would get the fundamental theorem of line integrals. As it is, it already does not make sense because the left hand side is a scalar and the right hand side is a vector field.

16) [✓ T] F There exists a vector field $\vec{F}$ and a scalar function $g(x,y,z)$ such that $\text{curl}(\vec{F}) = \text{grad}(g)$.

Solution:
Take $g = 2z$, $F = [-y, x, 0]$.

17) [✓ T] F If $\vec{F}$ is a vector field in space and S is a surface bounding a solid and the surface is oriented outwards, then $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

Solution:
This follows from the divergence theorem or from Stokes theorem.

18) T [✓ F] The solid defined by $x^2 + y^2 + z^2 \leq 9, z \leq \sqrt{x^2+y^2}$ has volume $\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^3 \rho^2 \sin(\phi)\, d\rho d\phi d\theta$.

Solution:
This is the volume in cylindrical coordinates.

19) T [✓ F] For any vector field $\vec{F} = [P, Q, R]$, we have $|\text{div}(\vec{F})| \leq |\text{curl}(\vec{F})|$.

Solution:
Any relation is possible. An example showing it to be false is $[x, y, z]$.

20) [✓ T] F If $\int_a^b \int_c^d f(x, y)\ dxdy = \int_c^d \int_a^b f(x, y) dxdy$ for all a, b, c, d, then $f(x, y) = f(y, x)$ for all x, y.

Solution:
If $f(x, y) \neq f(y, x)$, then it is also not the same when integrating over a small rectangle

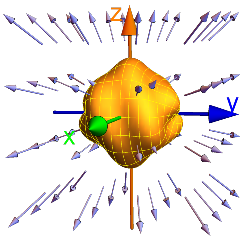
Problem 2) (10 points)

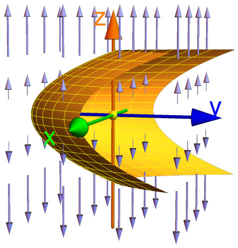
a) (2 points) We decide in two cases whether the **flux** of the vector field through the surface S is $\boxed{\text{positive}}$ or $\boxed{\text{negative}}$. In the left picture (belonging to the boxes to the left), S is oriented outwards, in the right picture S is oriented in the positive y direction.

The flux through the left surface is

Positive

Negative





The flux through the right surface is

Positive

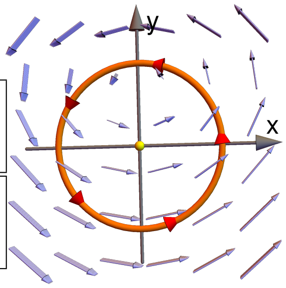
Negative

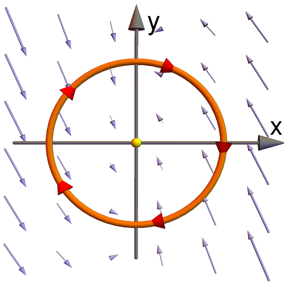
b) (2 points) Decide in each case, whether the **line integral** of the vector field along the closed circular loop is $\boxed{\text{positive}}$ or $\boxed{\text{negative}}$. Check the boxes on each side.

The line integral along the curve is

Positive

Negative





The line integral along the curve is

Positive

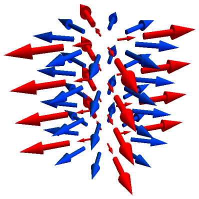
Negative

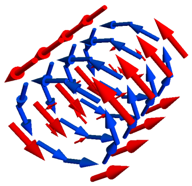
c) (2 points) Check on each side the box, for which the answer is positive $\boxed{> 0}$. Check exactly one box on the left for the left vector field and one box on the right for the right vector field.

For the field to the left: (check one box)

$|\text{curl}(\vec{F})| > 0$

$\text{div}(\vec{F}) > 0$





For the field to the right: (check one box)

$|\text{curl}(\vec{F})| > 0$

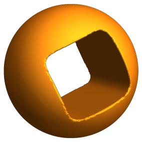
$\text{div}(\vec{F}) > 0$

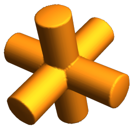
d) (2 points) Decide for each of the solids whether it is simply connected.

The solid to the left is (check one box)

simply connected

not simply connected





The solid to the right is (check one box)

simply connected

not simply connected

e) (2 points) Write down the names of two partial differential equations involving functions $f(t,x)$ for which only the first derivative with respect to t appear.

Solution:
a) Positive, negative
b) Positive, negative
c) div(F)¿0, —curl(F)—¿0
d) not simply connected, simply connected e) There are three PDEs we have seen: heat, Burger, transport

Problem 3) (10 points) No justifications are needed.

In this problem we repeat a routine task practiced already in the homework. The problem deals with the **three fundamental derivative operations** div, curl and grad defined in multivariable calculus. There are 27 ways to combine three such operations and being interested in combinatorics, we wonder how many operations are defined. To find out, we select in problems a) to c) the cases which are NOT defined.

a) (3 points) In the following table, cross out every expression which is NOT defined. Let $f(x, y, z)$ be a function of three variables.

$\text{grad}(\text{grad}(\text{grad}(f)))$	$\text{grad}(\text{curl}(\text{grad}(f)))$	$\text{grad}(\text{div}(\text{grad}(f)))$
$\text{curl}(\text{grad}(\text{grad}(f)))$	$\text{curl}(\text{curl}(\text{grad}(f)))$	$\text{curl}(\text{div}(\text{grad}(f)))$
$\text{div}(\text{grad}(\text{grad}(f)))$	$\text{div}(\text{curl}(\text{grad}(f)))$	$\text{div}(\text{div}(\text{grad}(f)))$

b) (3 points) In the following table, cross out every expression which is NOT defined. Let $\vec{F} = [P, Q, R]$ denote a vector field in $\mathbf{R}^3$.

$\text{grad}(\text{grad}(\text{curl}(\vec{F})))$	$\text{grad}(\text{curl}(\text{curl}(\vec{F})))$	$\text{grad}(\text{div}(\text{curl}(\vec{F})))$
$\text{curl}(\text{grad}(\text{curl}(\vec{F})))$	$\text{curl}(\text{curl}(\text{curl}(\vec{F})))$	$\text{curl}(\text{div}(\text{curl}(\vec{F})))$
$\text{div}(\text{grad}(\text{curl}(\vec{F})))$	$\text{div}(\text{curl}(\text{curl}(\vec{F})))$	$\text{div}(\text{div}(\text{curl}(\vec{F})))$

c) (3 points) In the following table, cross out every expression which is NOT defined. Again, $\vec{F} = [P, Q, R]$ is a vector field in $\mathbf{R}^3$.

$\text{grad}(\text{grad}(\text{div}(\vec{F})))$	$\text{grad}(\text{curl}(\text{div}(\vec{F})))$	$\text{grad}(\text{div}(\text{div}(\vec{F})))$
$\text{curl}(\text{grad}(\text{div}(\vec{F})))$	$\text{curl}(\text{curl}(\text{div}(\vec{F})))$	$\text{curl}(\text{div}(\text{div}(\vec{F})))$
$\text{div}(\text{grad}(\text{div}(\vec{F})))$	$\text{div}(\text{curl}(\text{div}(\vec{F})))$	$\text{div}(\text{div}(\text{div}(\vec{F})))$

d) (1 point) Two of the following expressions are always zero (either the zero number or zero vector). Which ones? As before, $\vec{F} = [P, Q, R]$ is a vector field and $f(x, y, z)$ a scalar function in $\mathbf{R}^3$

$\text{curl}(\text{grad}(f))$	$\text{curl}(\text{curl}(\vec{F}))$
$\text{div}(\text{grad}(f))$	$\text{div}(\text{curl}((\vec{F})$

Solution:
Note that grad grad, and grad curl, and curl div and div div are combinations which do not work. This allows to cross off many cases simultaneously like the first column in a), the first column in b) and the second and third column in c).

a) All except curl curl grad, div curl grad, grad,div grad

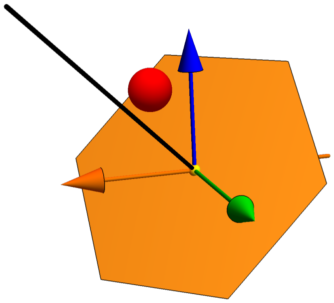
b) All except curl curl curl, div curl curl grad div curl

c) all except curl grad div and div grad div

d) curl grad =0 and div curl=0.

By the way, there are $F(k+3)$ different ways to write a meaningful sentence in the *div*, *grad*, *curl* language, where $F(k)$ is the k'th Fibonacci number. For $k=1$ there are 3, for $k=2$ there are 5, for $k=3$ (this was this exercise) there are 8.

Problem 4) (10 points)



a) (5 points) Find the distance of the point $P=(3,4,5)$ to the line $\vec{r}(t)=[t,t,t]$.

b) (5 points) Find the distance of the point $P=(3,4,5)$ to the plane $x+y+z=0$.

Solution:
Use the distance formulas.

a) $|[3,4,5]\times[1,1,1]|/|[1,1,1]| = \sqrt{6}/\sqrt{3} = \sqrt{2}$

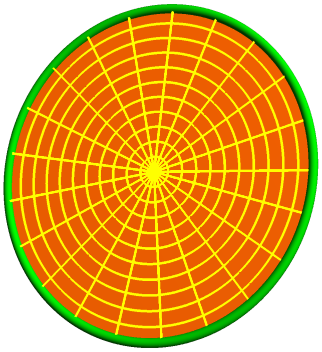
b) $|[3,4,5]\cdot[1,1,1]|/|[1,1,1]| = 12/\sqrt{3} = 4\sqrt{3}$.

Problem 5) (10 points)

a) (5 points) Find the surface area of the **spider web** surface parametrized by

$$\vec{r}(u,v) = [v\sqrt{2}\cos(u), v\sin(u), v\sin(u)]$$

with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.



b) (5 points) Find the arc length of the boundary curve

$$\vec{r}(t) = [\sqrt{2}\cos(t), \sin(t), \sin(t)]$$

with $0 \leq t \leq 2\pi$.

Solution:

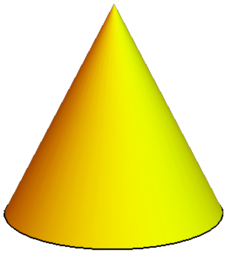
a) We have $r_u = [-v\sqrt{2}\sin(u), v\cos(u), 0]$ and $r_v = [\sqrt{2}\cos(u), \sin(u), \sin(u)]$. Now, $r_u \times r_v = [0, \sqrt{2}v, -\sqrt{2}v]$ which has length $2v$. The surface area is $\int_0^{2\pi}\int_0^1 2v \; dvdu = \boxed{2\pi}$.

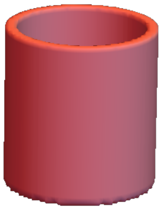
b) $\vec{r}\,'(t) = [-\sqrt{2}\sin(t), \cos(t), \cos(t)]$. Which has length $|\vec{r}'(t)| = \sqrt{2\sin^2(t) + 2\cos^2(t)} = \sqrt{2}$. So, the arc length is $\int_0^{2\pi} \sqrt{2} \; dt = \boxed{2\pi\sqrt{2}}$.

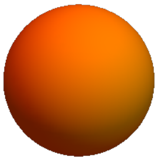
Problem 6) (10 points) No justifications are needed.

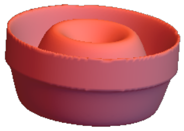
a) (5 points) We **teach an AI** to recognize objects and ask to match the pictures with the parametric surface. Do the job for the AI and match the parametrizations which best suit the surfaces. In each case, the parameters (u,v) range over some region R which are not given as it is irrelevant for the matching task here.

a) $\vec{r}(u,v) = [\sin(v)\cos(u), \sin(v)\sin(u), \cos(v)]$
b) $\vec{r}(u,v) = [u, v, \sin(u^2+v^2)]$
c) $\vec{r}(u,v) = [(1-u)\cos(v), (1-u)\sin(v), u]$
d) $\vec{r}(u,v) = [\cos(v), \sin(v), (1-u)]$
e) $\vec{r}(u,v) = [(1-u), v, 1]$



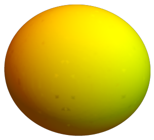
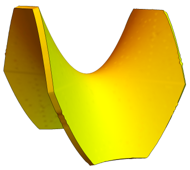
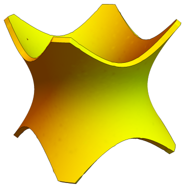
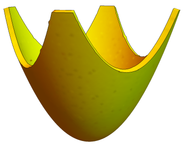
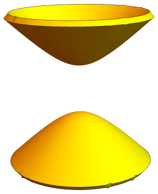






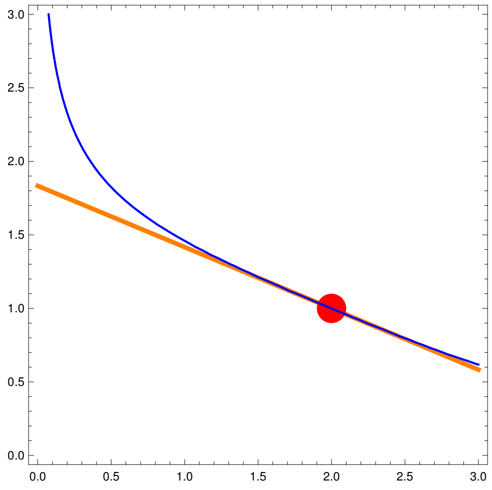
Enter the letters a)-e) into the boxes below the surfaces. There is an exact match.

b) (5 points) A **CAPTCHA** is a test designed to distinguish humans from AI's. The task is to recognize and label objects. To show that you are human, name the surfaces in one or two words each.

Solution:
a) c,d,a,e,b
b) sphere, hyperbolic paraboloid, 1-sheeted hyperboloid, paraboloid, 2-sheeted hyperboloid.

Problem 7) (10 points)



a) (2 points) Find the **tangent line** to the curve defined by

$$f(x,y) = x^2y + y^4x = 6$$

at the point $(2,1)$.
b) (2 points) Near $(2,1)$, the curve can be written as a **graph** $y = g(x)$. Find $g'(2)$.
c) (2 points) Find $L(x,y)$, the **linearization** of f at the point $(2,1)$.
d) (2 points) **Estimate** $f(2.001, 1.0001)$ using linearization.
e) (2 points) What is the **directional derivative** $D_{[3,4]/5}f(2,1)$?

Problem 8) (10 points)

a) (7 points) Classify all the critical points of the **1001 nights function**

$$f(x,y) = x^{1001} - 1001x + y^{1001} - 1001y$$

using the second derivative test.

b) (2 points, please justify briefly) Is there a global maximum of f on $x^2 + y^2 \leq 1$?

c) (1 point, no justification needed) Is there a global maximum of f on $\mathbf{R^2}$?

Solution:
This is a nice example because a computer algebra system chokes here (there are $1001 \cdot 1001$ complex solutions but only 4 real ones. You need the key word "Open Sesame!" to see the solution.
a) There are 4 critical points $(1,1),(-1,1),(1,-1),(-1,-1)$ which solve the equations $\nabla f(x,y) = 1001[x^{1000} - 1, y^{1000} - 1]$. The first is a minimum, the last is a maximum and the middle two are saddle points. b) The region is closed and bounded. Bolzano's theorem gives that there is a maximum.
c) No, there is no global maximum. Take $y = 0$ already to get a function which is unbounded.

Problem 9) (10 points)

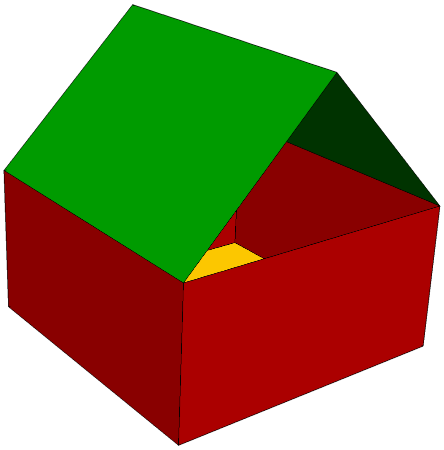
We want to minimize the paper used to build a **card house** of a given volume with 4 cards of size $x \times y$ and 3 cards of size $x \times x$. The area function

$$f(x,y) = 4xy + 3x^2$$

has a global minimum under the condition that the volume

$$g(x,y) = x^2y = 3/2$$

is fixed and $x > 0, y > 0$. Use Lagrange multipliers to find the minimal value $f(x,y)$.



Solution:
The gradients are $\nabla f = [4y_6x, 4x]$, $\nabla g = [2xy, x^2]$. The Lagrange equations are

$$\begin{aligned} 4y + 6x &= \lambda 2xy \\ 4x &= \lambda x^2 \\ 2x^2y = 3 & \end{aligned}$$

Eliminating λ from the first two equations gives $x = 2/3y$. Plugging into the third gives $x = 1, y = 3/2$.

Problem 10) (10 points)

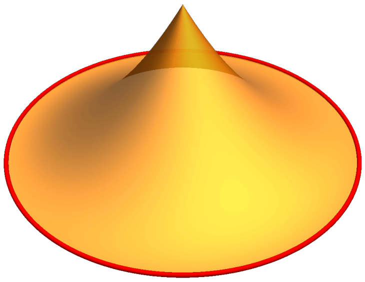
Find the flux of the curl of

$\vec{F}(x,y,z) = [-y(x+1)+z^7\cos(z), z, z^{21}+z\cos(xy)]$

through the surface S parametrized by

$\vec{r}(t,s) = [s\cos(t), s\sin(t), (1-s)^2], 0 \leq t \leq 2\pi, 0 \leq s \leq 1$.

The orientation of S is **downwards**.



Solution:
We use Stokes theorem. The boundary of the surface is parametrized by $\vec{r}(t) = [\cos(t), \sin(t), 0]$ but this gives an other orientation. We just change the sign at the end.

$$\int_0^{2\pi} [-\sin(t)(\cos(t)+1), 0] \cdot [-\sin(t), \cos(t), 0] \; dt = \pi \; .$$

Because of the downward orientation, the result is $\boxed{-\pi}$.

Problem 11) (10 points)

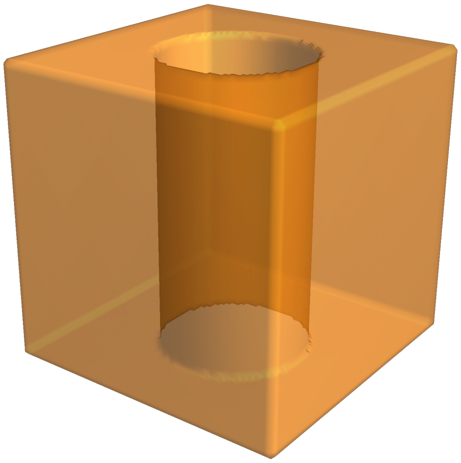
Find the flux $\iint_S \vec{F} \cdot d\vec{S}$ of the vector field

$\vec{F}(x,y,z) = [y^3 + \sin(y), z^3 + \sin(z), 5z + \sin(yx)]$

through the surface S bounding the region

$$x^2 \leq 4, y^2 \leq 4, z^2 \leq 4, x^2 + y^2 \geq 1 \ .$$

The surface S is oriented outwards.



Solution:
The divergence is 5. The result is 5 times the volume of the solid. The result is is $5(4^3 - 4\pi)$.

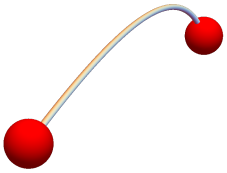
Problem 12) (10 points)

a) (4 points) Assuming $\vec{r}''(t) = [6,0,0]$ for all t and $\vec{r}(0) = [0,3,4]$ and $\vec{r}'(0) = [0,0,1]$, find $\vec{r}(1)$.

b) (6 points) Using the curve $\vec{r}(t)$ from a) and **using an integral theorem**, compute the line integral of the vector field

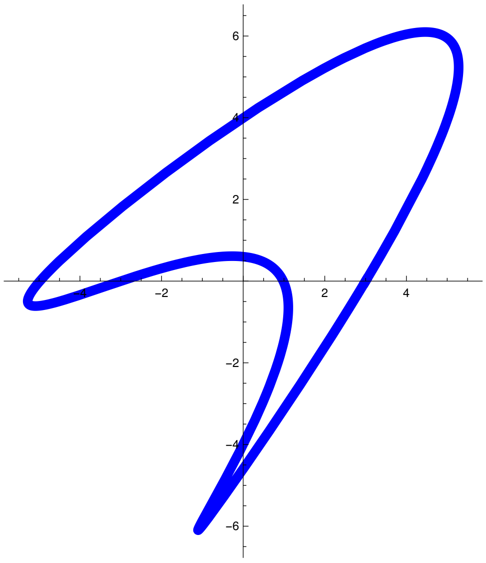
$$\vec{F}(x,y,z) = [\pi\cos(\pi x), 3y^2 + z, 4z^3 + y]$$

along the path $\vec{r}(t)$ from $t = 0$ to $t = 1$.



Solution:
a) Integrate twice and adjust the constant. We have $\vec{r}(t) = [3t^2, 3, 4 + t]$.
b) The vector field is a gradient field with potential $f(x,y,z) = x^4 + \sin(\pi x) + y^3 + z^4 + yz$. We just can evaluate f at the beginning $(0,3,4)$ and the end point $(3,3,5)$. We have $f(3,3,5) - f(0,3,4) = 5^4 - 4^4 + 3 = \boxed{372}$.

Problem 13) (10 points)



a) (5 points) Find the volume of the solid which is given in **cylindrical coordinates** (r, θ, z) as

$$1 \leq r \leq 3 + \cos(8\theta), 1 \leq z \leq 4 \ .$$

b) (5 points) Find the area of the **starfleet logo** bounded by

$$\vec{r}(t) = [3\cos(t) + 3\sin(2t), 4\sin(t) + 3\sin(2t)]$$

with $0 \leq t \leq 2\pi$. The region is illustrated to the right.

Solution:

a) The triple integral is

$$\int_0^{2\pi} \int_1^{3+\cos(8\theta)} \int_1^4 r \ dz dr d\theta$$

Starting with the inner integral we get the double integral

$$\int_0^{2\pi} \int_1^{3+\cos(8\theta)} 3r \ dr d\theta$$

Integrating next gives

$$\int_0^{2\pi} \frac{3(3+\cos(8\theta))^2}{2} - 3 \ d\theta$$

Now foil out and use that $\int_0^{2\pi} \cos^2(8\theta) \ d\theta = \pi$. The result is $(3/2)(9\dot{2}\pi + \pi) - 3\pi = \boxed{51\pi/2}$.

b) Use Green's theorem. We have to compute a line integral

$$\int_0^{2\pi} [0, 3\cos(t) + 3\sin(2t)] \cdot [-3\sin(t) + 6\cos(2t), 4\cos(t) + 6\cos(2t)] \ dt$$

When foiling out, there are 4 terms. The first term $\int_0^{2\pi} \cos^2(t) \ dt = \pi$ is non-zero. All the others are zero $\int_0^{2\pi} \cos(t)\cos(2t) \ dt$ $\int_0^{2\pi} \sin(2t)\cos(t) \ dt$ and $\int_0^{2\pi} \sin(2t)\cos(2t)$. One can see all this with double angle formulas like $\cos(t)\cos(2t) = \cos(t)(1 - 2\sin^2(t))$, $\int_0^{2\pi} 2\sin(t)\cos^2(t) \ dt$ and $\int_0^{2\pi} \sin(2t)\cos(2t) \ dt$ which all can be integrated by substitution to get zero. The final result is 12π.