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- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and nice unless stated otherwise.
- **Show your work**. Except for problems 1-3, we need to see details of your computation. If you are using a theorem for example, state the theorem.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F

The surface $-x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid.

Solution:
Look at the xy-trace.

2) T F

The vector projection $\vec{\text{proj}}_{\vec{w}}(\vec{v})$ of a vector $\vec{v}$ onto a non-zero vector $\vec{w}$ is always non-zero.

Solution:
The two vectors can be perpendicular

3) T F

The linearization of $f(x,y) = 5 + 7x + 3y$ at any point (a,b) is the function $L(x,y) = 5 + 7x + 3y$.

Solution:
A linear function has the function as a linearization.

4) T F

For any function $f(x,y,z)$, for any unit vector $\vec{u}$ and any point (x_0, y_0, z_0) we have $D_{\vec{u}}f(x_0, y_0, z_0) = -D_{-\vec{u}}f(x_0, y_0, z_0)$.

Solution:
Changing the direction of $\vec{u}$ changes the sign by definition $D_u f = \nabla f \cdot \vec{u}$.

5) T F

There is a vector field $\vec{F} = [P, Q, R]$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{F})$.

Solution:
The right hand side is a scalar while the left hand side is a vector.

6) T F

The formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$ holds if $\vec{v}, \vec{w}$ are vectors in space and α is the angle between them.

Solution:
Yes, this is an important formula for the length of the cross product.

7) T F

If $\vec{F} = [-2y, 2x]$ and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise, then $\int_C \vec{F} \cdot d\vec{r} = 16\pi$.

Solution:
Just write out the integrand $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = [-4\cos(t), 4\sin(t)] \cdot [-2\sin(t), 2\cos(t)] = 8\pi$. When integrated from 0 to 2π, we get 16π.

8) T F

The parametrization $\vec{r}(u, v) = [u, \sqrt{1 - u^2 - v^2}, v]$, $u^2 + v^2 \leq 1$ describes a half sphere.

Solution:
Indeed, the entries x, y, z satisfy $x^2 + y^2 + z^2 = 1$

9) T F

The vectors $\vec{v} = [1, 0, 1]$ and $\vec{w} = [-1, 1, 1]$ are perpendicular.

Solution:
Their dot product zero.

10) T F

If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) then $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C.

Solution:
It would be true if div were replaced by curl.

11) T F

The vector field $\vec{F} = [e^x, e^y, e^z]$ satisfies $\text{grad}(\text{div}(\vec{F})) = \vec{F}$.

Solution:
Indeed. Just compute it and use that $(e^x)' = e^x$.

12) T F

If $\vec{F}(x, y, z)$ has zero curl everywhere in space, then $\vec{F}$ is a gradient field.

Solution:
This is a result which follows from the Stokes theorem.

13) T F

If $\vec{r}(u,v)$ is a parametrization of the surface $g(x,y,z) = x^2 + e^{y^3} + z^2 = 5$ then for any u and v we have $\nabla g(\vec{r}(u,v)) \cdot \vec{r}_u(u,v) = 0$.

Solution:
This is true for any surface $g(x,y,z) = c$. The reason is that the gradient of g is perpendicular to the level surface and that $\vec{r}_u$ is tangent to the surface.

14) T F

The equation $\text{grad}(\text{div}(\text{grad}(f))) = [0,0,0]$ always holds.

Solution:
A simple example is $f(x,y,z) = x^3$.

15) T F

There is a non-constant function $f(x,y,z)$ such that $\text{grad}(f) = \text{curl}(\text{grad}(f))$ everywhere.

Solution:
The right hand side is 0. So, ∇f has to be constant.

16) T F

If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S oriented outwards is the volume of the enclosed solid.

Solution:
By the divergence theorem.

17) T F

If $f(x,y)$ is maximized at (a,b) under the constraint $g(x,y) = c$, then $\nabla f(a,b)$ and $\nabla g(a,b)$ are parallel.

Solution:
They indeed are by the Lagrange equations

18) T F

The distance between a point P and the line L through two different points A, B is given by the formula $|\vec{PA} \times \vec{AB}|/|\vec{PA}|$.

Solution:
It would be true if $\vec{PA}$ were replaced by $\vec{AB}$.

19) T | F — The unit tangent vector $\vec{T}(t)$ is always perpendicular to the vector $\vec{T}\,'(t)$.

Solution:
We have shown that in class. It follows from differentiating $T \cdot T = 1$.

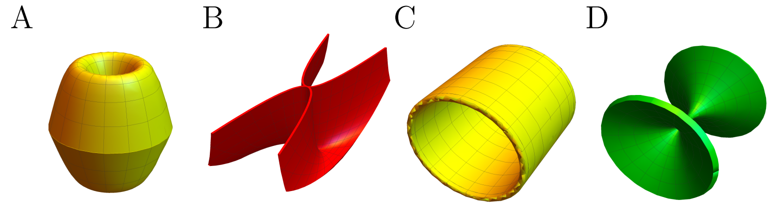
20) T | F — The vector field $\vec{F}(x, y, z) = [x^5, x^6, x^7]$ can not be the curl of another vector field.

Solution:
Its divergence is not zero. So that it can not happen

Problem 2) (10 points) No justifications are necessary.

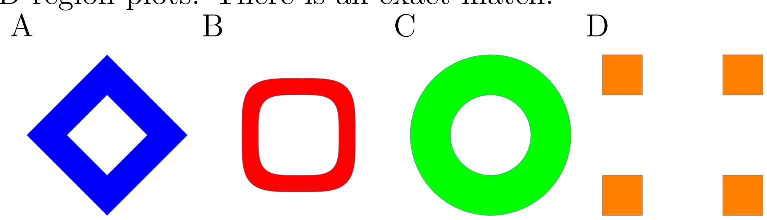
a) (2 points) Match the following surfaces. There is an exact match.

Surface $\vec{r}(u,v) =$	A-D
$[u, u^2 v, v^2]$	
$[u^2\cos(v), u, u^2\sin(v)]$	
$[\cos(v), \sin(u), \sin(v)]$	
$[v\cos(u), v\sin(u), \sin(v)]$	



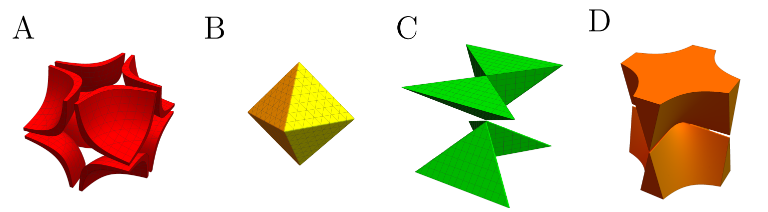
b) (2 points) Match the following 2D region plots. There is an exact match.

Region	A-D
$1 \leq x^2 \leq 4, 1 \leq y^2 \leq 4$	
$1 \leq x^2 + y^2 \leq 4$	
$1 \leq x^4 + y^4 \leq 4$	
$1 \leq \|x\| + \|y\| \leq 2$	



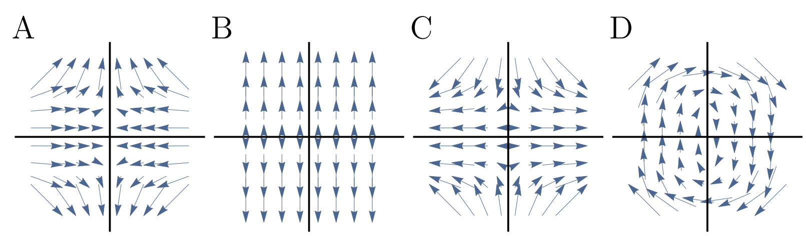
c) (2 points) Match the following 3D regions. There is an exact match.

Solid	A-D
$\|x\| + \|y\| + \|z\| \leq 2$	
$\|x\| < \|y\| < \|z\|$	
$1 < \|xyz\| < 2$	
$x^2y^2 < z^2$	



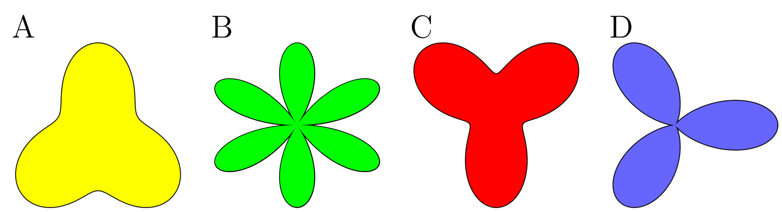
d) (2 points) The following figures display vector fields. There is an exact match.

Field	A-D
$\vec{F}(x,y) = [0, y]$	
$\vec{F}(x,y) = [-x, y^3]$	
$\vec{F}(x,y) = [y^3, -x]$	
$\vec{F}(x,y) = [x, -y^3]$	



e) (2 points) The following figures display polar regions. There is an exact match.

Polar region	A-D
$r \leq 1 + \cos(3\theta)$	
$r \leq 2 + \sin(3\theta)$	
$r \leq 3 - \sin(3\theta)$	
$r \leq \|\sin(3\theta)\|$	



Solution:
a) BDCA
b) DCBA
c) BCAD
d) BADC
e) DCAB

Problem 3) (10 points)

a) (6 points)

The concept of **boundary** plays an important role in integral theorems. In each of the following six rows, check exactly one entry which best describes the boundary.

The boundary of	solid	surface	curves	points	empty
$x^2 + y^2 + z^2 = 1$					
$x^2 + y^2 = 1, z = 0$					
$x^2 + y^2 + z^2 \leq 1, x = y = 0$					
$x^2 + y^2 + z^2 \leq 1$					
$x^2 + y^2 \leq 1, z = 0$					
$x^2 + y^2 = 1, z^2 \leq 1$					

b) (4 points) Match the following partial differential equations (PDEs) by picking 4 from the 5 given choices A-E.

PDE	Enter A-E
heat equation	
wave equation	
transport equation	
Burgers equation	

$u_{xx} = u_t$	A
$u_{xx} = u_t + uu_x$	B
$u_x = u_t$	C
$u_{xx} = -u_{tt}$	D
$u_{xx} = u_{tt}$	E

Solution:
a) empty, empty, points, surface, curves, curves.
Here is some more explanation:
The first is a closed surface which has no boundary.
The second is a closed curves which has no boundary.
The third is a line segment which has two points as a boundary.
The fourth is the unit ball which has the unit sphere, a surface, as a boundary.
The fifth is a closed disk, which has a circle, a curve as a boundary.
The sixth is a finite cylinder, which has two circles, two curves, as a boundary.
b) A,E,C,B.

Problem 4) (10 points)

a) (3 points) A surface S is parameterized by

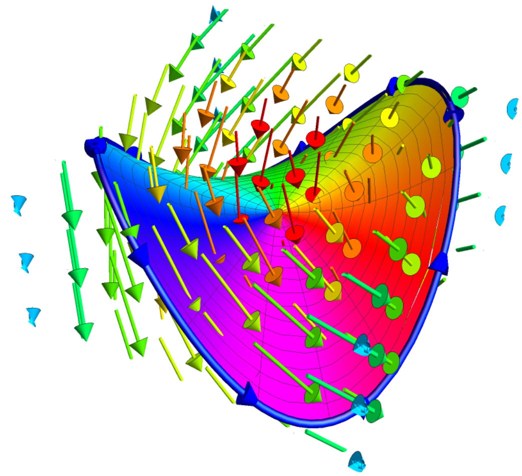
$$\vec{r}(u,v) = [u, v, uv] \ ,$$

where $u^2 + v^2 \leq 1$. Find its surface area.

b) (3 points) Parametrize the boundary curve C matching the orientation $\vec{r}_u \times \vec{r}_v$ of S, then compute the line integral $\int_C \vec{F} \cdot \ d\vec{r}$ with $\vec{F}(x,y,z) = [-y, x, 1]$.

c) (2 points) The coordinates of the surface S satisfy $xy - z = 0$. Find the tangent plane to S at $(2,1,2)$.

d) (2 points) Find the linearization $L(x,y)$ of $f(x,y) = xy$ at the point $(2,1)$.



Solution:
a) Since $|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2 + v^1 + 1}$ we have in polar coordinates $\int_0^{2\pi} \int_0^1 \sqrt{1+r^2} r \, dr d\theta = 2\pi/3(\sqrt{8} - 1)$.
b) The boundary curve is $\vec{r}(t) = [\cos(t), \sin(t), \cos(t)\sin(t)$. The line integral can be computed directly as $\vec{r}\,'(t) = [-\sin(t), \cos(t), \cos^2(t) - \sin^2(t)]$. which gives $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 2\cos^2(t)$ and the integral is, (use double angle formula) equal to $\boxed{2\pi}$.
P.S. Some have computed it with Stokes using $\text{curl}(\vec{F}) = [0,0,2]$ and $\vec{r}_u \times \vec{r}_v = [-v, -u, 1]$ as computed in a). We have to integrate the function 2 over the disk of radius 1. This is twice the area of the disk, which is again $\boxed{2\pi}$.
c) The gradient vector is $[y, x, -1]$ which is $[1, 2, -1]$ so that the equation of the plane is $x + 2y - z = d$. The constant can be obtained by plugging in the point. This is $\boxed{x + 2y - z = 2}$.
d) The gradient of f is $[y, x]$. At the point $(2,1)$ this is $[1,2]$. The linearization is
$\boxed{L(x,y) = f(2,1) + 1(x-2) + 2(y-1) = 2 + (x-2) + 2(y-1)}$

Problem 5) (10 points)

On November 17 2017, the **NASA Eagleworks paper** appeared, making the **EM drive** more probable. It might in future be used for deep space missions. The drive produces a thrust, apparently violating momentum conservation.

a) (5 points) Assume the drive flies in the gravitational field

$$\vec{F}(x,y,z) = [x^7 + xy^2z^2, x^2yz^2, x^2y^2z]$$

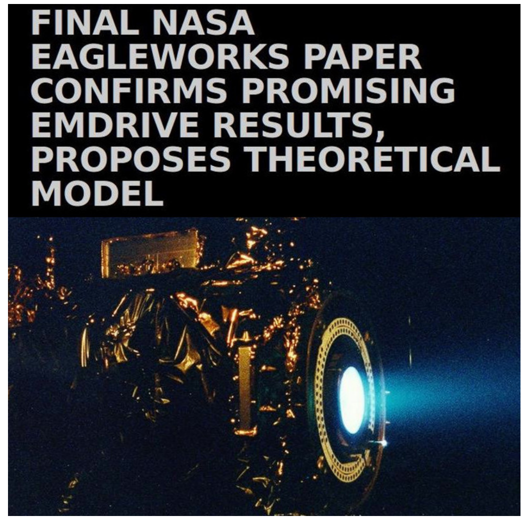
along the path

$$C : \vec{r}(t) = [t\cos(t), t\sin(t), t(5\pi - t)]$$

with $0 \leq t \leq 5\pi$. Compute the work

$$\int_0^{5\pi} \vec{F} \cdot d\vec{r}.$$

b) (3 points) Compute $d = |\int_0^{5\pi} \vec{r}\,'(t)\ dt|$.



c) (2 points) If L is the arc length of C, circle the one box below which applies:

$d = L$	$d > L$	$d < L$

Solution:
a) The field is conservative. There is a potential $f(x,y,z) = x^2y^2z^2/2 + x^8/8$. We can by the fundamental theorem of line integral, evaluate $f(-5\pi,0,0) - f(0,0,0) = (5\pi)^8/8$.
b) By the fundamental theorem of calculus, $\int \vec{r}\,'(t)\ dt = \vec{r}(5\pi) - \vec{r}(0) = [-5\pi, 0, 0]$. Its length is $\boxed{5\pi}$.
c) The arc length of the curve is longer than the shortest connection, the line which has been computed in b). Note that the arc length has the length $|\cdot|$ inside the integral. We have

$d = L$	$d > L$	$d < L$

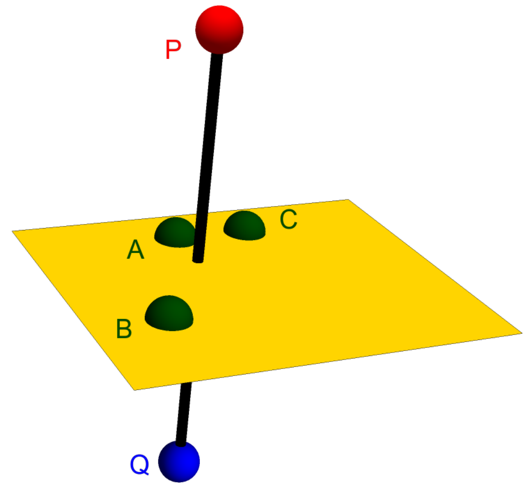
Problem 6) (10 points)

a) (2 points) Find the equation $ax+by+cz=d$ of the plane through $A=(1,1,1), B=(3,4,5), C=(4,4,2)$.

b) (3 points) Compute the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AC}$.

c) (3 points) Determine the volume of the parallelepiped spanned by $\vec{AB}, \vec{AC}, \vec{AP}$ where $P=(1,3,4)$.

d) (2 points) Find the distance $|\vec{PQ}|$, where Q is the mirror image of P opposite of the plane. It is determined by the fact that the middle point $(P+Q)/2$ is on the plane and that $\vec{PQ}$ is perpendicular to the plane.



Solution:
a) The normal vector is $\vec{n} = \vec{AB} \times \vec{AC} = [2,3,4] \cdot [3,3,1] = [-9,10,-3]$. The equation is $-9x+10y-3z=d$. The constant is obtained by plugging in a point. It is $\boxed{-9x+10y-3z=-2}$.

b) The area of the parallelogram is the length of the vector $\vec{n} = [-9,10,-3] = \boxed{\sqrt{190}}$.

c) The volume of the parallel epiped is $\vec{AP} \cdot \vec{n} = \boxed{11}$.

d) The distance is volume divided by area which is $11/\sqrt{190}$. The distance between P and Q is $\boxed{22/\sqrt{190}}$.

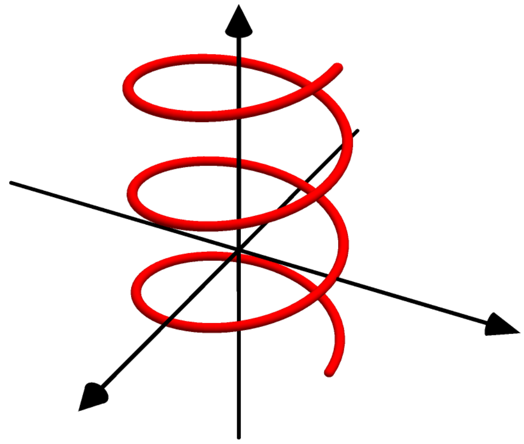
Problem 7) (10 points)

The triple scalar product is also written as

$$[\vec{u},\vec{v},\vec{w}] = \vec{u}\cdot(\vec{v}\times\vec{w}).$$

The **torsion** of a space curve is defined as

$$\frac{[\vec{r}\,',\vec{r}\,'',\vec{r}\,''']}{|\vec{r}\,'\times\vec{r}\,''|^2}.$$



a) (3 points) Compute $\vec{r}\,'(0),\vec{r}\,''(0),\vec{r}\,'''(0)$ for

$$\vec{r}(t) = [\cos(t),\sin(t),t].$$

b) (4 points) Compute the torsion of the curve at the point $\vec{r}(0)$.

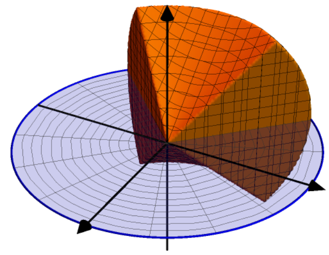
c) (3 points) Assume you have an arbitrary curve $\vec{r}(t)$ which is contained in the xy-plane. What is its torsion?

Solution:
a) $\vec{r}\,'(0) = [0,1,1]$.
$\vec{r}\,''(0) = [-1,0,0]$.
$\vec{r}\,'''(0) = [0,-1,0]$.
b) The triple scalar product is 1. The cross product $|\vec{r}\,'\times\vec{r}\,''|^2 = |[0,-1,1]|^2 = 2$. The torsion is $\boxed{1/2}$.
c) If $\vec{r}(t) = [x(t),y(t),0]$, then $\vec{r}\,'(t)\times\vec{r}\,'(t) = [0,0,x'y''-x''y']$. On the other hand, $\vec{r}\,'''(t) = [x'''(t),y'''(t),0]$. The dot product is $\boxed{\text{zero}}$.
P.S. Also more intuitive explanations worked: the three derivatives are all in the plane. They span a parallelepiped of zero volume. Therefore, the torsion, which has the volume in the nominator, is zero.

Problem 8) (10 points)

Let E be the solid

$$x^2 + y^2 \geq z^2, x^2 + y^2 + z^2 \leq 9, y \geq |x|.$$



a) (7 points) Integrate

$$\iiint_E x^2 + y^2 + z^2 \; dxdydz.$$

b) (3 points) Let $\vec{F}$ be a vector field

$$\vec{F} = [x^3, y^3, z^3].$$

Find the flux of $\vec{F}$ through the boundary surface of E, oriented outwards.

Solution:
a) The region is best described in **spherical coordinates**. The ϕ angle goes from $\pi/4$ to $3\pi/4$. The θ angle goes from $\pi/4$ to $3\pi/4$. The radius ρ goes from zero to 3. The integral is

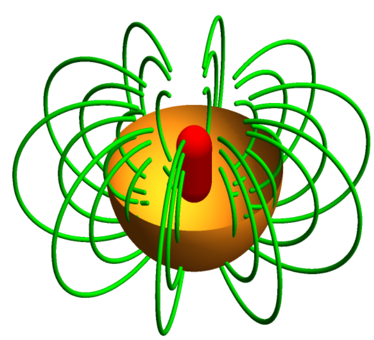
$$\int_{\pi/4}^{3\pi/4} \int_{\pi/4}^{3\pi/4} \int_0^3 \rho^2 \rho^2 \sin(\phi) \; d\rho d\phi d\theta \; .$$

The answer is $(\pi/2)(3^5/5)\sqrt{2} = \boxed{243\pi\sqrt{2}/10}$.
b) Since the divergence is $3x^2 + 3y^2 + 2z^2$ the result is just three times the result found in a). It is $\boxed{729\pi\sqrt{2}/10}$.

Problem 9) (10 points)

The vector field

$$\vec{A}(x, y, z) = \frac{[-y, x, 0]}{(x^2 + y^2 + z^2)^{3/2}}$$



is called the **vector potential** of the magnetic field

$$\vec{B} = \text{curl}(\vec{A}).$$

The picture shows some flow lines of this **magnetic dipole field** $\vec{B}$. Find the flux of $\vec{B}$ through the lower half sphere $x^2 + y^2 + z^2 = 1, z \leq 0$ oriented downwards.

Solution:
Since we have an integral of the curl of the vector field $\vec{A}$, we use **Stokes theorem** and integrate $\vec{A}(\vec{r}(t))$ along the boundary curve $\vec{r}(t) = [\cos(t), -\sin(t), 0]$. First of all, we have $\vec{A}(\vec{r}(t)) = [\sin(t), \cos(t), 0]$. The velocity is $\vec{r}\,'(t) = [-\sin(t), \cos(t), 0]$. The integral is $\int_0^{2\pi} -1\; dt = -2\pi$. The answer is $\boxed{-2\pi}$.

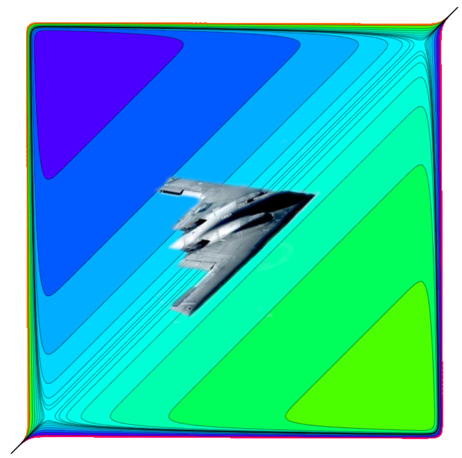
Problem 10) (10 points)

a) (8 points) Classify the critical points of the **area 51** function

$$f(x,y) = x^{51} - 51x - y^{51} + 51y$$

using the second derivative test. The reason why this function was chosen is classified.

b) (2 points) Does the function have a global maximum or global minimum on the region $x^2+y^2 \leq 1$ including the boundary? Write "yes" or "no" with a brief explanation. There is no need to find the global extrema.



Solution:
a) Setting the gradient $\nabla f = [51x^{50} - 51, 51y^{50} - 51]$ to $[0,0]$ gives the solutions $x = \pm 1, y = \pm 1$ and so the solutions $(1,1), (-1,1), (1,-1), (-1,-1)$. We have $f_{xx} = 50 \cdot 51 x^{49}$ and $f_{yy} = -50 \cdot 51 y^{49}$ and $f_{xy} = 0$ so that $D = f_{xx} f_{yy} - f_{xy}^2 = -(50 \cdot 51)^2 x^{49} y^{49}$. This is negative if the two coordinates have the same sign and positive else. Furthermore $f_{xx} > 0$ if x is positive. Therefore $\boxed{(1,1), (-1,-1) \text{ are saddle points}}$ and $\boxed{(-1,1) \text{ is the maximum}}$ and $\boxed{(1,-1) \text{ is the minimum}}$.
b) Yes by $\boxed{\textbf{Bolzano}}$, the continuous function f has both a maximum and minimum on the closed disc $x^2 + y^2 \leq 1$ as this region is both bounded and closed.

Problem 11) (10 points)

Using the Lagrange method, find the maximum and minimum of the elliptic curve function

$$f(x,y) = y^2 - x^3 - x^2 - x$$

on the circle $g(x,y) = x^2 + y^2 = 1$.

This problem is motivated from a real life application. To encrypt communication in **"WhatsApp"**, the elliptic curve 25519 given by $y^2 = x^3 + 486662x^2 + x$ over the prime $p = 2^{255} - 19 =$

57896044618658097711785492504343953926634992332820282019728792003956564819949

is used.

Solution:
The Lagrange equations are

$$\begin{aligned} -3x^2 - 2x - 1 &= \lambda(2x) \\ 2y &= \lambda 2y \\ x^2 + y^2 &= 1 \end{aligned}$$

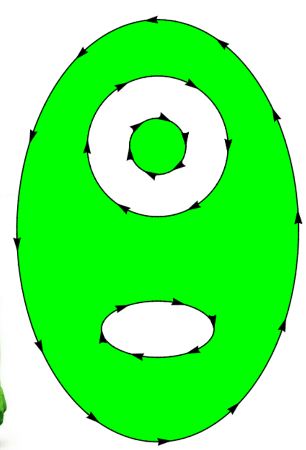
Eliminating λ gives $2y(-3x^2 - 2x - 1) = 2x2y$. The first possibility is that $y = 0$. In that case, $x = \pm 1$. If y is not zero, then we can divide it out and get $-3x^2 - 2x - 1 = 2x$ which is equivalent to $3x^2 + 4x + 1 = (3x+1)(x+1)$. We had already $x = -1$ so that the only other solution is $x = -1/3$ which gives $y = \pm\sqrt{8}/3$. Now we have all the critical points. $(1,0), (-1,0), (-1/3, \sqrt{8}/3), (-1/3, -\sqrt{8}/3)$. To find the maximum and minimum, we evaluate the function f at the critical points. The maximum is at $\boxed{(-1/3, \pm\sqrt{8}/3)}$. The minimum is at $\boxed{(1,0)}$.

Problem 12) (10 points)

Given the scalar function $f(x,y) = x^5 + xy^4$, compute the line integral of

$$\vec{F}(x,y) = [5y + 3y^2, 6xy + y^4] + \nabla f$$

along the boundary of the **Monster region** given in the picture. There are four boundary curves, oriented as shown in the picture: a large ellipse of area 16, two circles of area 1 and 2 as well as a small ellipse (the mouth) of area 3. The picture describes the orientations of the boundary curves perfectly and they are as they are! We warn you not to ask about this, or else we will bring in "Mike" from **Monsters, Inc.**



Solution:
The curl is -5. By **Green's theorem** we just have to compute the right areas. If the eye of Mike were oriented differently, we would get the area itself which is $16-(2-1)-3=12$ and the answer would be -60. However, since the inner eye is oriented wrong, that area is counted negatively. So instead of adding 1 we subtract 1 for that line integral contribution. The answer is $(-5)[16-2-3-1] = \boxed{-50}$.

Problem 13) (10 points)

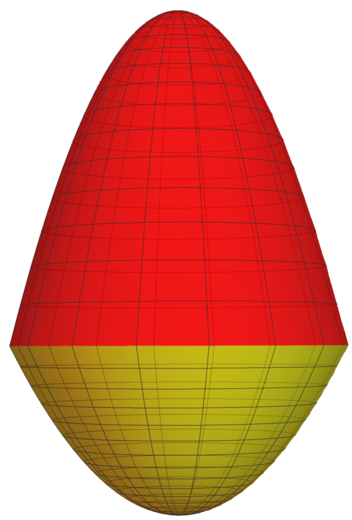
"ProtEgg" is a defense spell. It produces an egg shaped solid E enclosed by the surfaces

$$S : z = 2 - 2x^2 - 2y^2, z \geq 0 ,$$

where S is oriented upwards and

$$T : z = x^2 + y^2 - 1, z \leq 0 ,$$

where T is oriented downwards.



a) (4 points) Find the volume $\iiint_E 1\, dxdydz$ of E.

b) (4 points) The spell uses a force field

$$\vec{F}(x, y, z) = [0, 0, 2x^2 + 2y^2 + z] .$$

S is parametrized by $\vec{r}(u, v) = [u, v, 2 - 2u^2 - 2v^2]$ with $u^2 + v^2 \leq 1$ oriented upwards. Compute the flux $\iint_S \vec{F} \cdot d\vec{S}$ without an integral theorem.

c) (2 points) The flux $\iint_T \vec{F} \cdot d\vec{S}$ can be determined using an integral theorem. What is the value of the flux? Check all that apply:

A	B	C	D
$\iint_S \vec{F} \cdot d\vec{S}$	$-\iint_S \vec{F} \cdot d\vec{S}$	$\iiint_E 1\, dV + \iint_S \vec{F} \cdot d\vec{S}$	$\iiint_E 1\, dV - \iint_S \vec{F} \cdot d\vec{S}$

Solution:
a) The region E is described by $x^2 + y^2 \leq 1, \quad x^2 + y^2 - 1 \leq z \leq 2 - 2x^2 - 2y^2$ which is in polar coordinates:

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 1, \quad r^2 - 1 \leq z =\leq 2 - 2r^2.$$

$$\begin{aligned} \iiint_E 1\,dx\,dy\,dz &= \int_0^{2\pi} \int_0^1 \int_{r^2-1}^{2-2r^2} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^1 (3 - 3r^2) r\,dr\,d\theta = \frac{3\pi}{2}. \end{aligned}$$

The answer is $\boxed{\frac{3\pi}{2}}$.
b) Since $\vec{r}_u = [1, 0, -4u]$ and $\vec{r}_v = [0, 1, -4v]$, we have $\vec{r}_u \times \vec{r}_v = [4u, 4v, 1]$ and

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_{u^2+v^2 \leq 1} \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times = \vec{r}_v)\,du\,dv \\ &= \iint_{u^2+v^2 \leq 1} 2\,du\,dv = 2\pi. \end{aligned}$$

The answer is $\boxed{2\pi}$.
c) Since S and T bound the solid E, by the **divergence theorem** we have

$$\iint_S \vec{F} \cdot d\vec{S} + \iint_T \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\,dV = \iiint_E 1\,dV.$$

This implies that

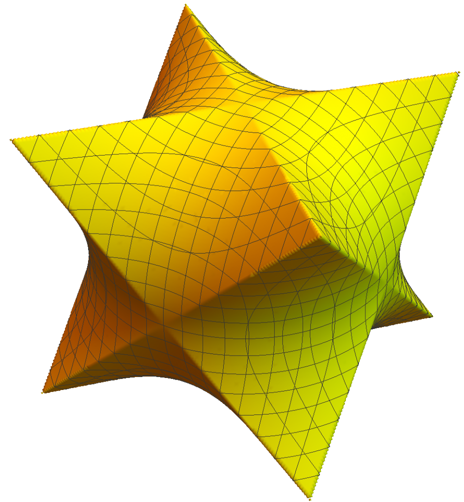
$$\iint_T \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\,dV = \iiint_E 1\,dV - \iint_S = \vec{F} \cdot d\vec{S}.$$

Only answer $\boxed{\text{D}}$ is correct.

Problem 14) (10 points)

Archimedes computed the volume of the intersection of three cylinders. The **Archimedes Revenge** is the problem to determine the volume V of the solid R defined by

$$x^2+y^2-z^2 \leq 1, y^2+z^2-x^2 \leq 1, z^2+x^2-y^2 \leq 1 .$$



Archimedes Revenge is brutal! It is definitely to hard for this exam. We give you therefore the volume $V = \log(256)$. Now to the **actual exam problem**: find the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of

$$\vec{F}(x,y,z) = [2x+y^2+z^2, x^2+2y+z^2, x^2+y^2+2z]$$

through the boundary surface S of R, assuming that S is oriented outwards.

Solution:
This is a problem for the divergence theorem.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{V})\, dV .$$

As the divergence is constant 6, the result will be 6 times the volume of the solid. It is $\iiint_E 6 dV = 6$ Volume(E) $= \boxed{6\log(256)}$.