

Homework 11: Partial derivatives

This homework is due Monday, 10/7.

- 1 If $f(x, y) = \sqrt{4 - x^2 + 4y^2}$, compute $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate this with either hand-drawn sketches of traces (functions of one variable) or using computer plots.

Solution:

$$f(x, y) = \sqrt{4 - x^2 - 4y^2} \Rightarrow f_x(x, y) = -x(4 - x^2 - 4y^2)^{-1/2}$$

and $f_y(x, y) = -4y(4 - x^2 - 4y^2)^{-1/2}$. $-1/\sqrt{19}$. $8/\sqrt{19}$.

- 2 Find the partial derivatives $f_x(x, y)$, $f_y(x, y)$ of the function $f(x, y) = 5x^y$ at the point $(1, 2)$. In order to do so, you first want to rewrite x^y using the exponential function.

Solution:

$$f(x, y) = 5x^y \Rightarrow f_x(x, y) = 5yx^{y-1}, f_y(x, y) = 5x^y \ln x$$

- 3 Find the first partial derivatives $f_x(x, y)$, $f_y(x, y)$ of the function

$$f(x, y) = \int_y^x \sin(t^2) dt .$$

Solution:

$$f(x, y) = \int_y^x \sin(t^2) dt \Rightarrow f_x(x, y) = \frac{\partial}{\partial x} \int_y^x \sin(t^2) dt = \sin(x^2) \text{ by the Fundamental Theorem of Calculus. } f_y(x, y) = \frac{\partial}{\partial y} \int_y^x \sin(t^2) dt = -\frac{\partial}{\partial y} \sin(t^2) = -\sin(y^2).$$

4 Verify that for any constant k and any constant α , the function

$$u(x, t) = e^{-\alpha^2 k^2 t} \sin(kx)$$

is a solution of the heat equation $u_t(x, t) = \alpha^2 u_{xx}(x, t)$.

Solution:

$$u = e^{-\alpha^2 k^2 t} \sin(kx) \Rightarrow u_x = k e^{-\alpha^2 k^2 t} \cos(kx), u_{xx} = -k^2 e^{-\alpha^2 k^2 t} \sin(kx), \text{ and } u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin(kx). \text{ Thus } \alpha^2 u_{xx} = u_t$$

5 Verify that the function

$$u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

is a solution of the three dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

Solution:

$$u = 1/\sqrt{x^2 + y^2 + z^2} \Rightarrow u_x = (-\frac{1}{2})(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2} \text{ and } u_{xx} = -(x^2 + y^2 + z^2)^{-3/2} - x(-\frac{3}{2})(x^2 + y^2 + z^2)^{-5/2}(2x) = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}. \text{ By symmetry, we can see}$$

$$\text{that } u_{yy} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \text{ and } u_{zz} = \frac{2z^2 - y^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

$$\text{Thus } \frac{u_{xx} + u_{yy} + u_{zz}}{(x^2 + y^2 + z^2)^{5/2}} = \frac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - y^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}} = 0.$$

Main definitions

If $f(x, y)$ is a function of two variables, then $\frac{\partial}{\partial x}f(x, y)$ is defined as the derivative of the function $g(x) = f(x, y)$, where y is considered a constant. It is called **partial derivative** of f with respect to x . The partial derivative with respect to y is defined similarly. We also write $f_x(x, y) = \frac{\partial}{\partial x}f(x, y)$ and $f_{yx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}f$.

Clairaut's theorem If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

We will in the next lecture look more at partial differential equations: an equation for an unknown function $f(x, y)$ which involves partial derivatives for at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, one calls it an **ordinary differential equation**. Both ordinary and partial differential equations are of enormous importance in other sciences. Here are examples we are going to look in the next class:

- 1 The **wave equation** $f_{tt}(t, x) = f_{xx}(t, x)$ governs the motion of light or sound.
- 2 The **heat equation** $f_t(t, x) = f_{xx}(t, x)$ describes diffusion of heat or spread of an epidemic.
- 3 The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane.
- 4 The **transport equation** $f_t = f_x$ is used to model transport in a wire.