Homework 16: Directional Derivatives

This homework is due Monday, 10/21.

1 a) Find the gradient of $f(x,y,z) = \sin(\pi(x + 11yz))$ at the point $P = (1,3,5)$. b) Use the gradient to find the rate of change of f at P in the direction of the vector $\vec{u} = [2/7, 3/7, 6/7]$.
c) Find a direction of the form $\vec{v} = [a,b,c]$ in which the directional derivative of f at $(1,3,5)$ is $10\sqrt{2}\pi$.

2 a) (5 points) Find the directional derivative of the function $f(x,y) = \log(4 + x^2 + 3y^2)$ at the point $P = (2,1)$ in the direction of the vector $\vec{v} = [-1,2]/\sqrt{5}$. (We use the notation $\log = \ln$).
b) (5 points) Find the directional derivative of $f(x,y,z) = xy + yz + zx$ at the point $P = (1,-1,3)$ in the direction from P to $Q = (2,4,5)$. (Take the unit vector!).

3 a) Find the direction of steepest descent for $f(x,y,z) = \frac{(x+y)}{z}$ at the point $P = (1,1,-1)$.
b) Find the value of the rate of change at $(1,1,-1)$ in that direction found in a).

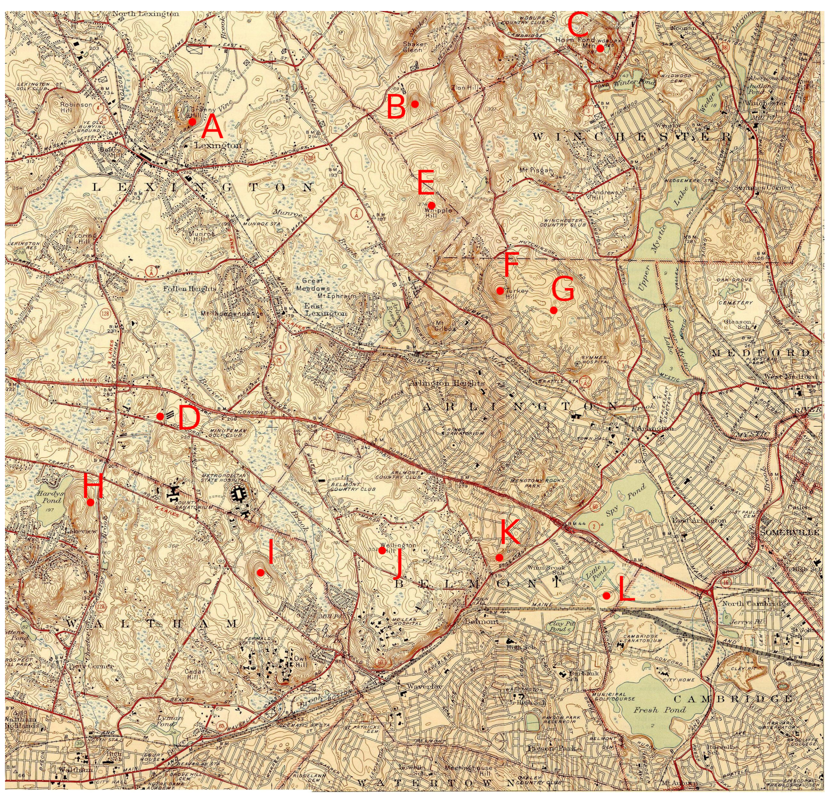
4 You know that a function $f(x,y,z)$ satisfies $f(0,0,0) = 33$ and $D_{[1,1,1]/\sqrt{3}}f(0,0,0) = 4/\sqrt{3}$ and $D_{[1,1,0]/\sqrt{2}}f(0,0,0) = 7/\sqrt{2}$ and $D_{[1,2,2]/3}f(0,0,0) = 12$. Estimate $f(0.01, -0.001, 0.1)$ using linearization!

5 On the course website there is a map of Cambridge,Arlington, Lexington, Winchester from 1946. The map features level curves of the height function $f(x,y)$ as well as points $A - K$ marked in red.
a) At which of the points A-L are all directional derivative $D_v f$

zero and all second directional derivatives $D_v(D_v f)$ negative?
b) Where are all $D_v f$ zero and $D_v D_v f$ take both positive and negative values.
c) At which points is $f_x = 0$ and $f_y > 0$. d) At which of the points is $|\nabla f|$ maximal?

Main definition:

If $\vec{v}$ is a unit vector then $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ is the **directional derivative** of f in the direction $\vec{v}$. For $\vec{v} = \nabla f / |\nabla f|$, the directional derivative is $D_v f = \nabla f \cdot \nabla f / |\nabla f| = |\nabla f|$ so that f **increases** in the direction of the gradient. The value $|\nabla f|$ is the maximal slope.



Geographic data visualization

This is an extra credit problem.

The following table shows elevation data near the town of Waltham in Massachusetts. Each grid is 4000 feet apart and elevations are in feet. Let $f(x, y)$ be the elevation data assigned to position cell (x, y). The bottom left cell corresponds to the position $(1, 1)$, the top right cell to position $(9, 7)$.

269	246	213	253	240	253	233	236	253
226	210	203	213	220	246	256	213	240
269	203	187	233	236	230	259	203	210
233	180	223	269	112	180	194	157	128
138	164	203	161	69	59	59	75	105
177	112	174	98	69	69	46	46	49
233	171	85	52	72	82	59	56	56

a) Sketch the contour $f(x, y) = 200$ by drawing out any division line between two cells where the value changes from below to above 200. You can do this on a printout of this worksheet.
b) If you move from cell $(1, 1)$ to cell $(7, 7)$ on a straight line, you see the elevation change in time. Draw a relief graph of that height.
c) Between which points is the slope of the landscape described in the table maximal? How steep do you estimate it to be there? In what direction would one need to walk to go straight uphill? (you can restrict to up/down/left/right). d) Mathematica which provided the table does not give details on how these numbers were obtained. Speculate about the choices which were made when collecting these elevation measurements? There

is no right, nor wrong answer, but we want you to think of at least two different possibilities which were used to write down the data points.

e) The Mathematica below gives example code to visualize elevation data. Most towns in the US are covered. Chose a town of your liking different from Cambridge visualize its geographic height data. You can get a Mathematica notebook from the website, run the modified notebook and print it out.

Mathematica example code with real Geo data:

```
A = Reverse[Normal[ GeoElevationData[
     Entity["City", {"Cambridge", "Massachusetts", "UnitedStates"}],
     GeoProjection -> Automatic]]];
ReliefPlot[A]
ListDensityPlot[A]
ListPlot3D[A]
ListContourPlot[A]
```