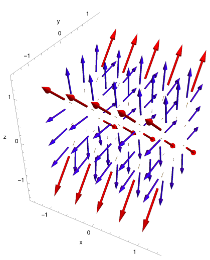
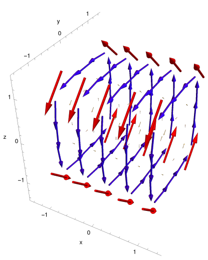
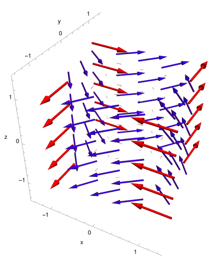
Homework 25: Vector fields

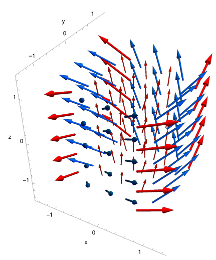
This homework is due Wednesday, 11/12

1 Match the vector fields $\vec{F}$ with the plots labeled A-D.
a) $\vec{F}(x,y,z) = [y^3 - x^3, y^3 + x^3, 0]$, b) $\vec{F}(x,y,z) = [0, -z^3, y^3]$
c) $\vec{F}(x,y,z) = [x^3 - y^3, x^3 + y^3, 1]$, d) $\vec{F}(x,y,z) = [0, y^3, z^3]$


A


B


C

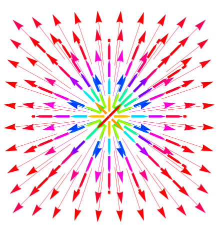
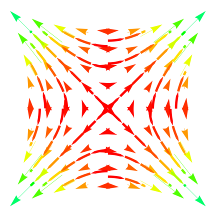
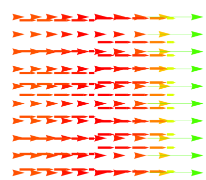
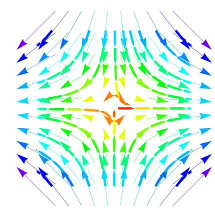

D

Solution:
Look for the zeros. If the z-component is zero, then the field leaves the planes $z = const$ invariant. If the x-component is zero, then the field leaves the planes x=const invariant.
a) C)
b) B)
c) D)
d) A)

2 a) Compute the gradient vector field $\vec{F} = \nabla f$, where $f(x,y,z) = 1/(x^2 + y^2 + z^2)$.
b) Find the gradient vector field for $f(x,y) = \log(x^2 + y^2)/2$.
c) Given the vector field $\vec{F} = [P,Q] = [\frac{4x}{\sqrt{4x^2+21y^2}} + 6x^2y + 1, 2x^3 + \frac{21y}{\sqrt{4x^2+21y^2}}]$. Check that $Q_x - P_y = 0$ and find a function $f(x,y)$ for which $\nabla f = \vec{F}$.

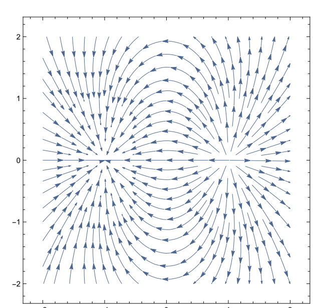
Solution:
a) $\nabla f = [-\frac{2x}{(x^2+y^2+z^2)^2}, -\frac{2y}{(x^2+y^2+z^2)^2}, -\frac{2z}{(x^2+y^2+z^2)^2}]$. b) $[x, y]/(x^2 + y^2)$.
c) $f(x, y, z) = \sqrt{x^2 + 7y^2} + 2x^3y + x$.

3 Match the functions f with the plots of their gradient fields labeled $A - D$. Give reasons for your choices.
a) $f(x, y) = x^2 - y^2$ b) $f(x, y) = x^2y^2$
c) $f(x, y) = \log(x^2 + y^2 + 1)$ d) $f(x, y) = e^{x^2 \sin(x)}$

A  B  C  D 

Solution:
A c)
B b)
C d)
D a)

4 a) Sketch the vector field $\vec{F}(x, y) = [2x, 4y]$ and then sketch some flow lines. What shape to these flow lines appear to have? Find in particular the flow line $\vec{r}(t)$ with $\vec{r}(0) = [1, 1]$.



b) Find a function f such that the vector field ∇f looks as in the picture above.

Solution:
a) Write $x' = 2x, y' = 4y$ to get $x(t) = Ae^{2t}$ and $y(t) = Be^{4t}$. We see that $x^2 = cy$ so that we have parabolas. b) The field to $f_1 = 1/((x+1)^2 + y^2)$ is a vector field which points towards the point $(-1, 0)$. The field $f_2 = -1/((x-1)^2 + y^2)$ is a vector field which points away from the point $(1, 0)$. If we want to have both things combined, just add the potentials: the function $f(x, y) = 1/((x+1)^2 + y^2) - 1/((x-1)^2 + y^2)$ should work.

5 a) Plot $\vec{F}(x, y) = [x^3 + y^4, x^2 + x^5]$ using Mathematica for $-2 < x < 2, -2 < y < 2$.
b) Make a stream plot of the field $\vec{F}(x, y) = [x^3 - \sin(y) + 2y, \sin(x^5) + x^2]$ using Mathematica. If you start on the line $y = -1$, there is a watershed threshold so that if x is larger than this value the flow will go to the right and to the left, the path will go to the left. Find this value (round to the next integer).

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StreamPlot[{x+y,x^2},{x,-2,2},{y,-2,2}]
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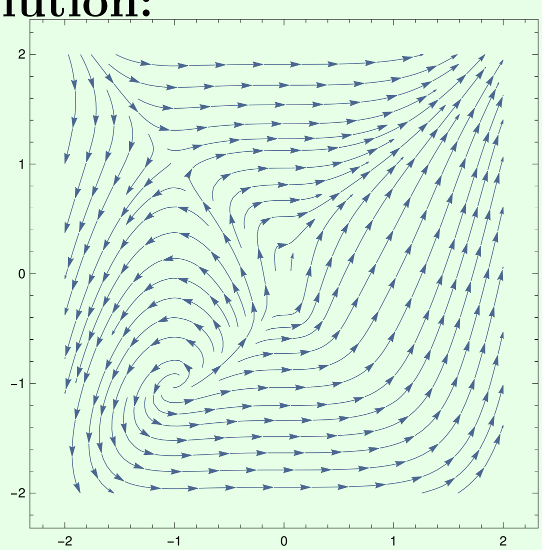
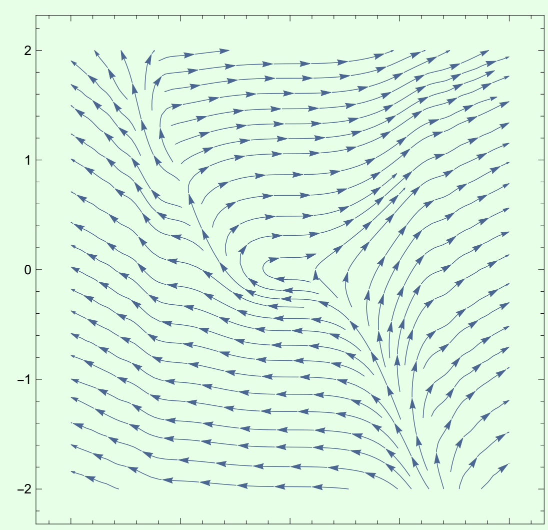
Solution:

a) 

b) 

The watershed is near x=1.

Main definitions:

A **vector field** assigns to each point (x, y) a vector $\vec{F}(x, y) = [P(x, y), Q(x, y)]$. In space, a vector field has three components $\vec{F}(x, y, z) = [P(x, y, z), Q(x, y, z), R(x, y, z)]$. Vector fields of the form $\vec{F}(x, y) = [P, Q] = \nabla f(x, y)$ or $\vec{F}(x, y, z) = [P, Q, R] = \nabla f(x, y, z)$ are called **gradient fields**. The function f is called the potential of F and can be found integration. The **flow line** of $\vec{F}$ is a curve $\vec{r}(t)$ for which $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$. If the field is a velocity field of a river, then $r(\vec{t})$ is the path a particle follows.