MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 4: Partial Derivatives

Partial Derivatives

If we differentiate a function with respect to one variable, keeping the other variables constant, we call this a **partial derivative**. For example, if we differentiate $f(x, y)$ with respect to x, we write $\frac{\partial}{\partial x} f(x, y)$ or simply $f_x(x, y)$. For example if $f(x, y) = x^3y^5$, then $f_x(x, y) = 3x^2y^5$.

We can also differentiate multiple times. For example, f_{xx} means that we differentiate twice with respect to x. Or f_{xy} means that we differentiate first with respect to x and then with respect to y. And f_{yx} means that we differentiate first with respect to y and then with respect to x. For example if $f(x, y) = x^3y^5$, then $f_{xx}(x, y) = 6xy^5$ and $f_{xy}(x, y) = 15x^2y^4$.

Clairaut

Fortunately, we do not have to keep track about the order of differentiation. The reason is that there is a theorem about partial derivatives called **Clairaut's Theorem**:

If f_{xy} and f_{yx} are both continuous functions, then

$$\boxed{f_{xy} = f_{yx}}$$

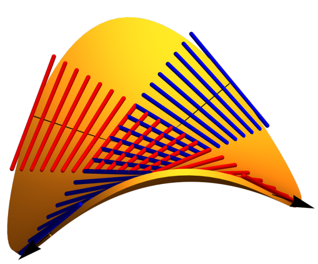

Figure 1. A visualization of Clairaut's theorem.

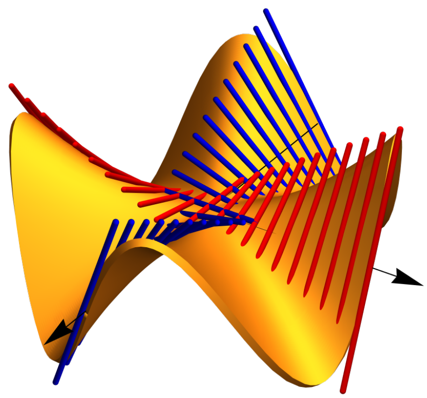

FIGURE 2. If f_{xy} and f_{yx} are not continuous, then Clairaut can fail.

THE MEANING

The value $f_x(x, y)$ is the slope you see when you slice the graph in the x direction. The value $f_y(x, y)$ is the slope you see when you slice the graph in the y direction.

> $f_x(x, y)$ is the slope of a x-trace

Single variable calculus also tells us that $f_{xx}(x, y)$ is positive if the x-trace is concave up and negative if the x-trace is concave down.

> $f_{xx}(x, y)$ is the concavity of the x-trace

More tricky is the interpretation of $f_{xy}(x, y)$. Imagine a stick pointing in the x direction tangent to the surface. This stick has slope $f_x(x, y)$ Now move the stick perpendicular into the y direction. The slope changes. If $f_{xy}(x, y) > 0$, the slope of the stick gets larger when moving in the y direction.

> $f_{xy}(x, y)$ is a torsion, the change of the x-slope when moving in the y direction.

Imagine you fly in the y direction with the right wing is in the x direction and you roll counter clockwise then the surface traced by the wing has $f_{xy} > 0$. Now, if we fly in the x direction and the right wing is in the y direction we still roll counter clockwise.

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