MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 6: Computing integrals

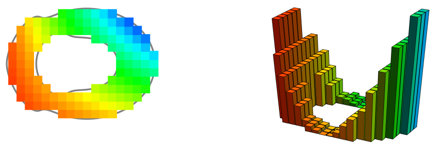

Figure 1. $\iint_R f(x,y)\, dA$ is a signed volume.

Area between horizontal curves

For two type of regions the integral $\iint_R f(x,y)\, dA$ can be computed nicely. The first is the region sandwiched between two graphs $g(x)$ and $h(x)$. The area between two horizontal graphs $g(x) \leq h(x)$ is

$$\int_a^b \int_{g(x)}^{h(x)} 1 \; dydx \; .$$

Evaluating the inner integral gives $\int_a^b h(x) - g(x)\; dx$.

Integral between horizontal curves

More generally, we have the double integral

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y) \; dydx \; .$$

First compute the inner integral $\int_{g(x)}^{h(x)} f(x,y)\; dy$ then the second.

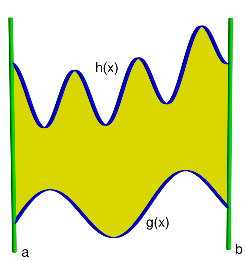

Figure 2. A region sandwiched between horizontal curves.

Area between vertical curves

If we replace x and y, we get the area between vertical curves:

$$\int_a^b \int_{g(y)}^{h(y)} 1 \; dxdy \; .$$

Evaluating the inner integral gives $\int_a^b h(y) - g(y) \; dy$. We add up the lengths of vertical slices of width dy when slicing at y

Integral between vertical curves

Replacing 1 with a more general $f(x, y)$ we get

$$\int_a^b \int_{g(y)}^{h(y)} f(x, y) \; dxdy \; .$$

First integrate $\int_{g(y)}^{h(y)} f(x, y) \; dx$, which is the integral over a horizontal slice, then integrate over y.

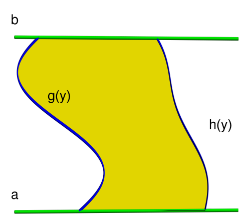

FIGURE 3. A region sandwiched between vertical curves.

Horizontal or Vertical Slices?

A) What is $\iint_R x^2 y dA$ if the region R between $y = 1 - 2x, y = 1 + 2x, x = 0$.

B) What is $\iint_R e^y \; dA$ if the region R is bound by $x = 0, y = x$ and $y = 1$.

C) What is $\iint_R \frac{1}{\sqrt{1-y^2}} \; dA$ if R is the unit disc $R = \{x^2 + y^2 \leq 1\}$.

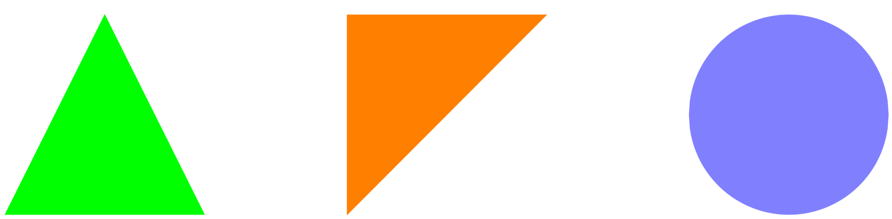

FIGURE 4. The regions in A),B) and C)

Section Oliver Knill, knill@math.harvard.edu, Harvard College, Fall 2022