MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 7: Polar integrals

The integration factor

When using polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$, the area element $dA = dxdy$ becomes the area element $rdrd\theta$. There is an **integration factor** $\boxed{r}$ due to the fact that sectors close to the origin have area smaller than sectors further away. Indeed the proportionality factor is r. There are various ways to show this. Later we will learn about the cross product, you might learn some linear algebra and see it as a determinant. But it can also be seen with a picture: a small square $d\theta dr$ in the (r, θ) plane is mapped by $T : (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a **sector segment** S in the (x, y) plane which has area $r\ drd\theta$. [1]

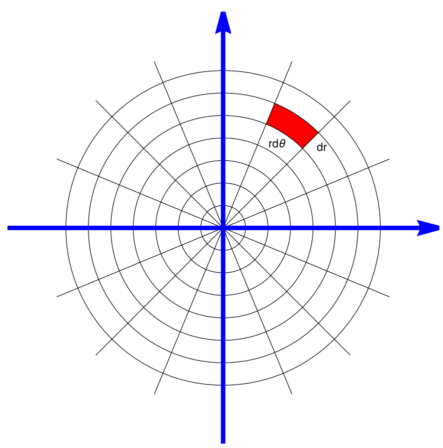

FIGURE 1. A sector in distance r of dimension $d\theta$ and dr has area $dA = rdrd\theta$.

Area of the circle

The most important example is the computation of the area of a circle of radius L. In Cartesian coordinates this already requires trig substitution

$$\int_{-L}^{L} \int_{-\sqrt{L^2-x^2}}^{\sqrt{L^2-x^2}} 1\ dydx = 2\int_{-L}^{L} \sqrt{L^2 - x^2}\ dx$$

[1] The error $\frac{dr^2}{2} d\theta/2$ is of smaller order

as we have seen last time. In polar coordinates, this is much easier: the area of a circle of radius L is

$$\iint_R 1 \; dA = \int_0^{2\pi} \int_0^L 1 \cdot r dr d\theta = \pi L^2 \; .$$

4. Moment of inertia of the circle

The number $\int_R x^2 + y^2 \; dA$ is the **moment of inertia** of the region R with respect to the origin. Use polar coordinates using $\boxed{r^2 = x^2 + y^2}$.

$$\int_0^{2\pi} \int_0^L r^2 \; r dr d\theta = \frac{L^4}{4} 2\pi = \frac{L^4 \pi}{2} \; .$$

The gifted problem

The anti derivative of e^{-x^2} can not be expressed using elementary functions like sin, cos, exp, log or polynomials. Still, we can compute

$$I = \int_{-\infty}^{\infty} e^{-x^2} \; dx \; .$$

The idea is one of the most astounding in calculus: use the fact that

$$I^2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2} \; dx dy$$

and that the later integral can be computed in polar coordinates

$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \; dr d\theta \; .$$

This can be evaluated as π. Therefore, the original integral I is $\sqrt{\pi}$. In probability theory, we say $e^{-x^2}/\sqrt{\pi}$ is a **probability density function** PDF. In order to have the variance 1 one usually scales this and calls

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

the **standard normal distribution**.

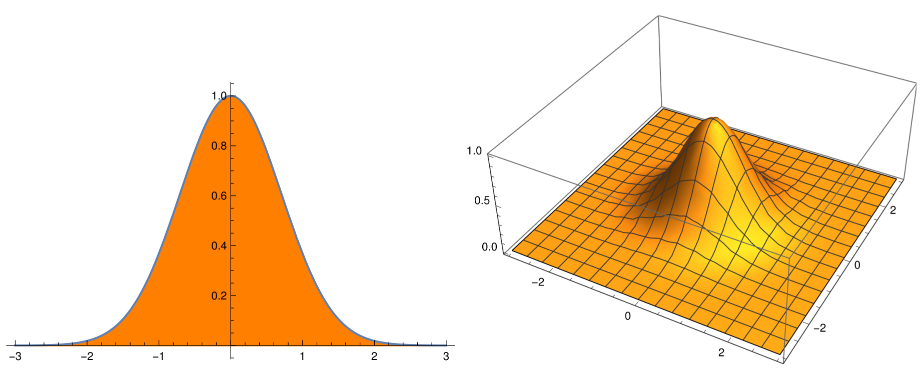

Figure 2. A bell curve e^{-x^2} and a two dimensional analog $e^{-x^2-y^2}$. We have seen $\int_{\mathbb{R}} e^{-x^2} \; dx = \sqrt{\pi}$ and $\iint_{\mathbb{R}^2} e^{-x^2-y^2} \; dA = \pi$.