MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 8: Triple integrals

Triple integrals

In two dimensions, we have cut a region R into pieces of size dA then summed up $\int\int_R f(x,y)dA$. In three dimensions, we cut a solid U into pieces of volume dV then sum up $\int\int\int_U f(x,y,z)\ dV$. Think of dV as a little cube and $f(x,y,z)$ as some density or charge quantity. The integral sums up over all these values. There are two methods. The burger method slicing into two dimensional slices will be discussed in next hour. The fries method reduces the triple integral to a double integral.

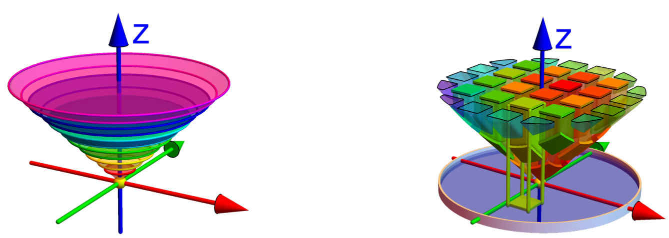

Figure 1. Burgers method and fries method. In this lecture, we look at the fries method $\int\int_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz] dA$ which produces a double integral after solving the inner integral. The integral on the right would be $\int\int_R \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\ dz\ dA$ where R is the unit disc $x^2+y^2 \leq 1$. You have to locate the bottom $z = \sqrt{x^2+y^2}$ (cone) and the top $z = 1$ (disk).

Typical setup

Most of the time, we compute integrals by writing the solid U sandwiched between two graphs $g(x,y)$ and $h(x,y)$ of functions of two variables defined on some domain R. For example for the sphere $x^2+y^2+z^2 = 1$ we can see the region as the unit disk R and the lower function to be $-\sqrt{1-x^2-y^2}$ and the upper function to be $\sqrt{1-x^2+y^2}$.

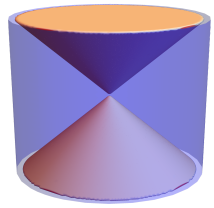

FIGURE 2. Typically, a solid is bound below from some surface and bound above from some other surface. In the situation of the figure, it is $\iint_R \int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} f(x,y,z)\; dz\; dA$, where R is a disk.

Volume and average

Similarly as $\iint_R 1\; dA$ was area, $\iiint_U 1\; dV$ is now a **volume**. Volume computations were historically the first triple integrals which appeared. If we have a function $f(x,y,z)$, for which we want to find the **average value** over a solid U, we compute

$$\frac{\iiint_U f(x,y,z)\; dV}{\iiint_U 1\; dV} .$$

Multi-dimensional integrals appear often in **multi-variate statistics**.

The Sphere

Archimedes was able to compute the volume of solids like the sphere using a comparison method. Here is how we do it in calculus. Let us compute the volume of the sphere using the fries method. The integral is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\; dzdydx .$$

After solving the most inner integral, we get

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2} dydx .$$

Now we have a double integral $\iint_R 2\sqrt{1-x^2-y^2} dA$ which is best computed using **polar coordinates**. We get $\int_0^{2\pi} \int_0^1 \sqrt{1-r^2} r\; drd\theta = 2\pi - \frac{1}{3}(1-r^2)^{3/2}|_0^1 = 4\pi/3$.

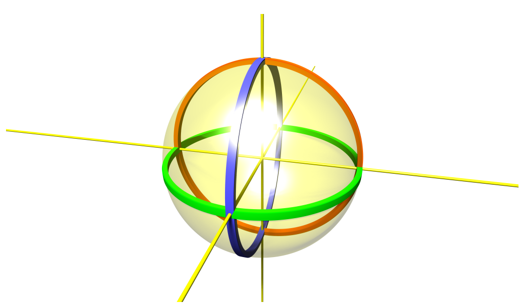

FIGURE 3. Computation of the sphere volume using calculus we know already.