MULTIVARIABLE CALCULUS

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Lecture 9: Triple integrals II

Review fries method

This is a second lecture on triple integral. We first review the reduction to double integral where the most inner integral is perpendicular to the **projection** R of the solid. This produced an integral in the form

$$\iint_R \int_{g(x,y)}^{h(x,y)} f(x,y,z)\; dzdA \; .$$

For example, let us compute the volume of the solid obtained by intersecting two **solid parabolas** $z > -2 + x^2 + y^2$ and $z < 6 - x^2 - y^2$. We get

$$\iiint_U 1 \; dV = \iint_R \int_{-2+x^2+y^2}^{6-x^2-y^2} 1 dzdA$$

The projection is the disc of radius 2 in the xy-plane. The radius of R is obtained by equating $-2 + x^2 + y^2 = 6 - x^2 - y^2$ which gives $2x^2 + 2y^2 = 8$ or $x^2 + y^2 = 4$.

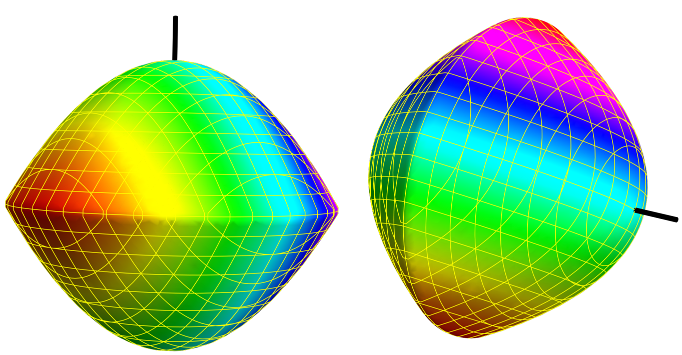

Figure 1. A solid obtained by intersecting two solid parabolas. We see two cases. It is first turned to be rotationally symmetric with respect to the z axis, in the second case it is rotationally symmetric with respect to the y axis.

Sometimes, an other axis of rotation is present. If we integrate $f(x,y,z) = x^2 + z^2$ over the solid given by the inequalities $y > -2 + x^2 + z^2$ and $y < 6 - x^2 - z^2$. The y

axes is a symmetry now, the projection is given by a disc of radius 2 in the xz-plane. We can write

$$\iint_R \int_{-2+x^2+z^2}^{6-x^2-z^2} x^2 + z^2 \, dydA$$

which becomes $\iint_R (8 - 2x^2 - 2z^2)(x^2 + z^2)\, dA$. Evaluation in polar coordinates gives $256\pi/15$.

The burger method

Example: To integrate over a **pyramid** U defined as $0 \leq z \leq 2 - |x| - |y| - |z|$, it is better to slice horizontally. The outer integral as a single integral over z which ranges from 0 to 2. If $R(z)$ is the slice at height z, which is a square $|x| + |y| < 2 - |z|$ of side length $\sqrt{2}(2 - z)$

$$\int_0^2 \iint_{R(z)} f(x, y, z)\, dA\, dz$$

To get the volume of the pyramid for example, we have

$$\int_0^2 \iint_{R(z)} 1\, dA\, dz = \int_0^2 2(2 - z)^2\, dz = \frac{16}{3}$$

which we know also from the fact that the base area is $A = 8$, the height is $h = 2$ and that $V = Ah/3 = 16/3$.

Example: The **octahedron** $|x| + |y| + |z| < 1$ consists of two pyramids stacked on top of each other. The z-range is obtained by putting $x = y = 0$. The projection region R is obtained by putting $z = 0$, so that we have $R = \{|z| + |y| < 1\}$ which is a square containing the points $(1, 0), (0, 1), (-1, 0), (0, -1)$. The area of the square is 2 and the height is 1 so that the volume of the pyramid is $2/3$ so that the octahedron has volume $4/3$. To write this as an integral, take the upper part only and multiply by 2. We get

$$2\int_0^1 \iint_{R(z)} 1\, dA\, dz = 2\int_0^1 2(1 - z)^2\, dz = \frac{4}{3}\ .$$

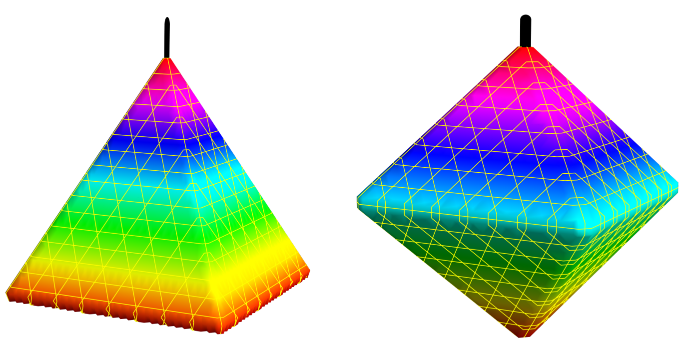

Figure 2. The first solid is a pyramid given as $0 \leq z \leq 2 - |x| - |y| - |z|$. Its projection is $|x| + |y| \leq 2$. Its volume is $8/3$. The second solid is the octahedron $|x| + |y| + |z| < 1$. Its volume is $4/3$.