OLIVER KNILL, MATH 21A

Lecture 10: Spherical Integrals

Spherical Wedges

When integrating in spherical coordinates, we need to know the volume of a spherical wedge at position (ρ, ϕ, θ) of size $d\rho, d\phi$ and $d\theta$.

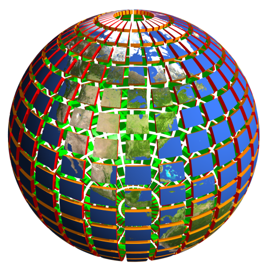

Figure 1. The earth mantle cut up into small spherical wedges. We see that near the poles where $\sin(\phi)$ is small the surface areas of the wedges is small. This picture was rendered in Povray.

Distortion factor

A small spherical wedge has side length $r d\theta = \rho \sin(\phi) d\theta$ and $\rho d\phi$ and $d\rho$. The volume therefore is

$$\boxed{dV = \rho^2 \sin(\phi) d\rho d\phi d\theta}$$

You can remember this also by keeping in mind that $dS = \rho^2 \sin(\phi) d\phi d\theta$ is the spherical area.

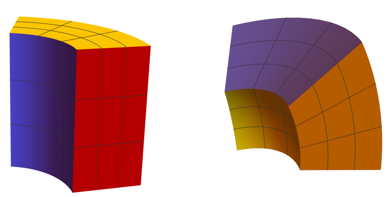

Figure 2. Cylindrical and spherical wedge. In the cylindrical case, the volume is $dV = r dz dr d\theta$. In the spherical case, the volume is $dV = \rho^2 \sin(\phi) d\rho d\phi d\theta$.

Example Spherical regions

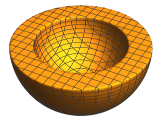

Figure 3. Half an avocado with stone removed. $4 < x^2 + y^2 + z^2 < 9, z < 0$.

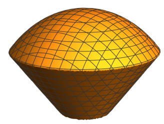

Figure 4. A muffin $1 \leq x^2 + y^2 + z^2 < 9, x^2 + y^2 < z^2$.

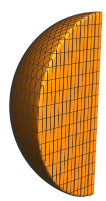

Figure 5. A slice of cheese $x^2 + y^2 + z^2 < 4, 0 \leq \theta \leq \pi/3$.

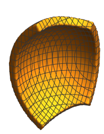

Figure 6. Part of an orange peel $7 < x^2 + y^2 + z^2 < 9, \pi/2 \leq \theta \leq \pi$. $x^2 + y^2 > z^2$.