MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 12: Dot product and Planes

Dot product

The **dot product** $\vec{v} \cdot \vec{w}$ between two vectors like $\vec{v} = \langle 2,3,4 \rangle$ and $\vec{w} = \langle 1,1,2 \rangle$ is the sum of the products of the components. It is in this case $\vec{v} \cdot \vec{w} = 2+3+8 = 13$. In general,

$$\boxed{\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3}$$

The dot product is useful already to get **magnitude** because $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$. We can compute like with numbers like $\|\vec{v} - \vec{w}\|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} - 2\vec{v} \cdot \vec{w} = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\vec{v} \cdot \vec{w}$.

Angles

If $\|\vec{w}\| = 1$ we can with $a = \vec{v} \cdot \vec{w}$ look at $\|\vec{v} - a\vec{w}\|^2 = (\vec{v} - a\vec{w}) \cdot (\vec{v} - a\vec{w}) = \|\vec{v}\|^2 + a^2\|\vec{w}\|^2 - 2a\vec{v} \cdot \vec{w} = \|\vec{v}\|^2 - a^2$ showing that $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|$. For a general $\vec{w}$ we get from this

$$\boxed{|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \|\vec{w}\|}$$

which is called the **Cauchy-Schwarz inequality**. This shows that $-1 \leq \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \leq 1$. There is therefore a unique $0 \leq \alpha \leq \pi$ such that $\cos(\alpha) = \frac{\vec{w} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$. We have proven

$$\boxed{\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos(\alpha)}$$

If $\vec{v} \cdot \vec{w} = 0$ the angle is called a **right angle**. If $\vec{v} \cdot \vec{w} > 0$, it is called an **acute angle**, if $\vec{v} \cdot \vec{w} < 0$ it is an **obtuse angle**.

Pythagoras

If we consider in a triangle ABC the vectors $\vec{v} = \vec{AB}$, $\vec{w} = \vec{AC}$ and $\vec{BC} = \vec{v} - \vec{w}$, we denote their lengths by a, b, c we have $a^2 = \|\vec{v}\|^2, b^2 = \|\vec{w}\|^2$ and $c^2 = \|\vec{v} - \vec{w}\|^2$ then we can combine the computations in the two previous paragraphs as

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos(\alpha)}$$

This is called the **Al Khashi formula**. A special case is when $\alpha = \pi/2$ which means $\cos(\alpha) = 0$. This is called the **Pythagorean identity**

$$\boxed{c^2 = a^2 + b^2}$$

Note that we have here a full proof of the Pythagorean identity. The dot product allowed us to do that.

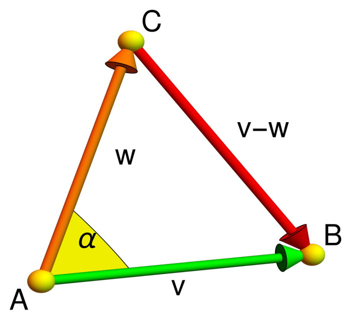

FIGURE 1. A triangle ABC defines three vectors $\vec{v}, \vec{w}, \vec{v} - \vec{w}$ an angle α. Using the dot product, we can derive the cos-formula of Al Khashi and Pythagoras.

SCALAR COMPONENT

The **scalar component** of a vector $\vec{v}$ onto a vector $\vec{w}$ is defined as

$$\boxed{\text{comp}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}.}$$

For example $\text{comp}_{\vec{v}}\vec{v} = \|\vec{v}\|$. This is a much more convenient way to compute the scalar component. We have struggled with it in the last lecture by giving only a vague definition. Note that $\text{comp}_{\vec{w}}\vec{v} = \|\vec{v}\|\cos(\alpha)$, where α is the angle between $\vec{v}$ and $\vec{w}$. The component does not depend on the length of $\vec{w}$.

PLANES

Given a vector $\vec{n} = \langle a, b, c \rangle$ and a point $P = (x_0, y_0, z_0)$ we can look at the collection of points $X = (x, y, z)$ such that $(X - P) \cdot \vec{n} = 0$. Writing this out means

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \ .$$

We can write this **plane** also as

$$\boxed{ax + by + cz = d}$$

where d is the constant obtained by plugging in the point (x_0, y_0, z_0) into $ax + by + cz$.

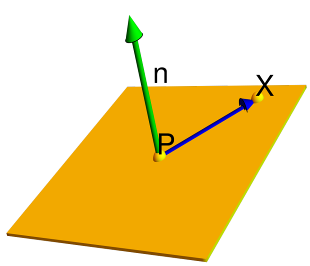

FIGURE 2. A **plane** $ax + by + cz = d$ through a point P perpendicular to $\langle a, b, c \rangle$.