MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 13: Cross product

Cross product

The **cross product** $\vec{v} \times \vec{w}$ between two vectors like $\vec{v} = \langle 2, 3, 4 \rangle$ and $\vec{w} = \langle 1, 1, 2 \rangle$ is a new vector. In this case $\vec{v} \times \vec{w} = \langle 2, 0, -1 \rangle$. The definition is

$$\boxed{\vec{v} \cdot \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle}$$

To compute this effectively, you can for example write the two vectors above each other (see class). The cross product is useful because $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$. You can directly show this by taking the dot product $\vec{v} \cdot (\vec{v} \times \vec{w})$ and check that it is zero. The product is anti-commutative $\vec{i} \times \vec{j} = -\vec{j} \times \vec{i}$ and not associative. For example, $(\vec{i} \times \vec{i}) \times \vec{j}$ is zero $\vec{0}$ but $\vec{i} \times (\vec{i} \times \vec{j}) = -\vec{j}$ is not the zero vector $\vec{0}$.

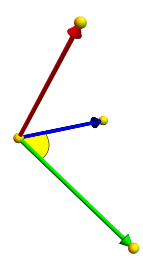

FIGURE 1. The cross product of two vectors is a vector perpendicular to both. Its length is $\|\vec{v}\| \|\vec{w}\| \sin(\alpha)$ where α is the angle of $\vec{v}$ and $\vec{w}$.

Magnitude

The **magnitude** of the cross product is:

$$\boxed{\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin(\alpha)}$$

This is very important because we can interpret this as the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$. To verify the length formula, one can use the **Cauchy-Binet** formula identity

$$\boxed{\|\vec{v} \times \vec{w}\|^2 + \|\vec{v} \cdot \vec{w}\|^2 = \|\vec{v}\|^2 \|\vec{w}\|^2}$$

Together with $|\vec{v} \cdot \vec{w}|^2 = \|\vec{v}\|^2 \|\vec{w}\|^2 \cos^2(\alpha)$ this gives the length formula for the cross product. The Cauchy-Binet formula can be checked directly.

Geometric use

Two important applications for the cross product are: 1) the **computation of the area** of a triangle. 2) getting the **equation of a plane** through three points:

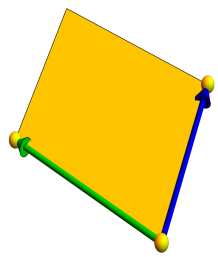

Figure 2. The length of the cross product is the area of the parallelogram spanned by the two vectors.

Problem: Let $A = (0,0,1), B = (1,1,1)$ and $C = (3,4,5)$ be three points in space $\mathbb{R}^3$. Find the equation of the plane through ABC and find the area of the triangle ABC.

Solution: Form $\vec{v} = \vec{AB} = \langle 1,1,0 \rangle$ and $\vec{w} = \vec{AC} = \langle 3,4,4 \rangle$. The vector $\vec{n} = \vec{v} \times \vec{w} = \langle 4,-4,1 \rangle$ gives us immediately the equation $4x - 4y + z = d$. We can plug in one of the points like $A = (x,y,z) = (0,0,1)$ to get the constant d. It is 1. The equation is

$$4x - 4y + z = 1 \ .$$

The triangle area is half of the area of the parallelogram: $\sqrt{16+16+1}/2 = \sqrt{33}/2$.

Applications

- the **Lorentz force** is $q\vec{v} \times \vec{B}$, where $\vec{v}$ is a velocity vector of a particle like an electron and $\vec{B}$ is the magnetic field and q is the charge.
- the **Coriolis force** $-2m\vec{\omega} \times \vec{v}$, where $\vec{\omega}$ is an angular velocity vector and $\vec{v}$ the particle velocity and m is the mass.
- The **torque** in physics is the vector $\vec{T} = \vec{r} \times \vec{F}$, where $\vec{F}$ is the force acting on a position $\vec{r}$. The vector $\vec{T}$ points in the direction of the axis of rotation.

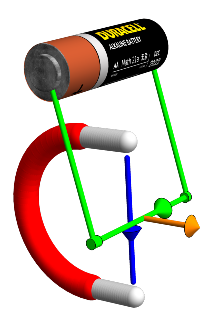

Figure 3. The Lorentz force is the cross product of the velocity of a charged particle and the magnetic field.