MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 16: Surfaces

Parametric Surfaces

A vector-valued function

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

from a region R in $\mathbb{R}^2$ to space $\mathbb{R}^3$ is called a **parametrized surface**. We distinguish the **parametrization** map from $\mathbb{R}^2$ to $\mathbb{R}^3$ and the picture of the surface. Think of u and v as coordinates on the surface. For fixed u or v, we get **grid curves**, seen in the graphics. The velocity $\vec{r}_u$ is tangent to the curve. Similarly, the velocity $\vec{r}_v$ is tangent to the curve. The vector $\vec{r}_u \times \vec{r}_v$ is perpendicular to the surface.

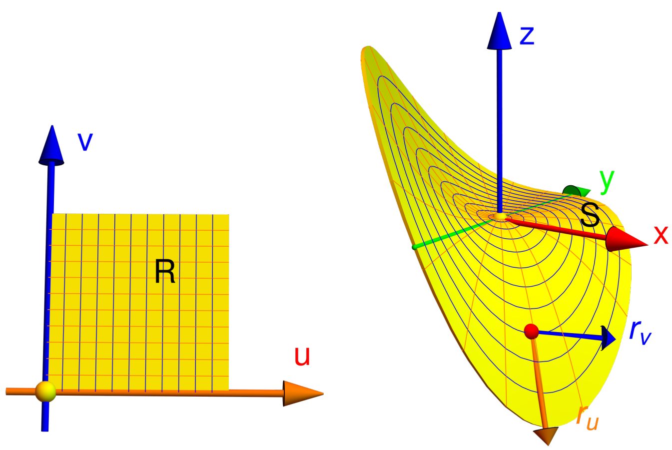

Figure 1. A parametrization of a surface defines a coordinates system on the surface.

Four basic examples

The sphere is parametrized

$$\vec{r}(\theta,\phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle \ .$$

The entire sphere is covered with $0 \le \theta < 2\pi, 0 \le \phi \le \pi$. By restricting to a part of this rectangle, we can get parts of the sphere.

Problem: Parametrize $(x-2)^2/16 + (y+1)^2/25 + z^2/9 = 1$. **Solution:** $\vec{r}(\theta,\phi)) = \langle 2 + 4\sin(\phi)\cos(\theta), -1 + 5\sin(\phi)\sin(\theta), 3\cos(\phi) \rangle$.

A plane can be parametrized when knowing two vectors in the plane.

$$\vec{r}(s,t) = \vec{P} + t\vec{v} + s\vec{w} \ .$$

Problem: Parametrize $x+2y+3z=6$. **Solution:** pick $\langle -2,1,0\rangle, \langle -3,0,1\rangle$ and point $P=(6,0,0)$ to get $\vec{r}(s,t) = \langle 6,0,0\rangle + t\langle -2,1,0\rangle + s\langle -3,0,1\rangle$.

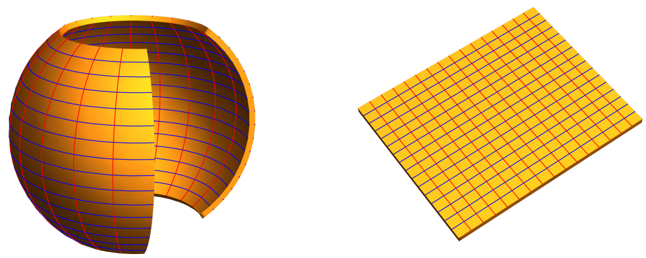

FIGURE 2. A parametrization of a sphere and a parametrization of a plane.

A graph $z = f(x,y)$ is parametrized by

$$\vec{r}(u,v) = \langle u, v, f(u,v)\rangle \ .$$

Problem: Parametrize the hyperbolic paraboloid $z = x^2 - y^2$ with $x^2 + y^2 \leq 1$.
Solution:

$$\vec{r}(u,v) = \langle u, v, u^2 - v^2\rangle$$

with $u^2 + v^2 \leq 1$. Also the above plane could be parametrized as a graph because $= (6 - x - 2y)/3$. We have $\vec{r}(u,v) = \langle u, v, (6-u-2v)/3\rangle$.
A surface of revolution $r = g(z)$ is parametrized by

$$\vec{r}(\theta, z) = \langle g(z)\cos(\theta), g(z)\sin(\theta), z\rangle \ .$$

Problem: Parametrize the hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution:
$\vec{r}(\theta, z) = \langle \sqrt{1+z^2}\cos(\theta), \sqrt{1+z^2}\sin(\theta), z\rangle$.

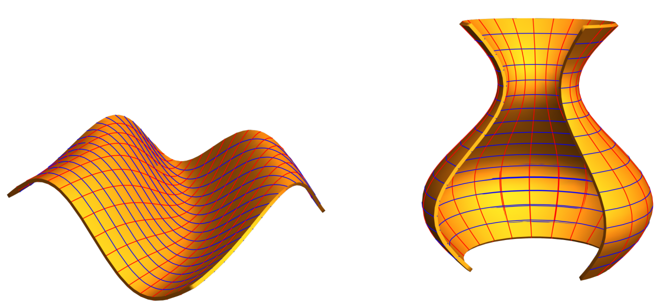

FIGURE 3. A parametrization of a graph and a surface of revolution.