MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 18: Surface integrals

Surface area

If $\vec{r}(u,v)$ is a parametrization of a surface, then $\vec{r}_u$ and $\vec{r}_v$ are tangent to the surface. Think of $\vec{r}_u du$ and $\vec{r}_v dv$ as sides of a small parallelogram of area $dS = |\vec{r}_u \times \vec{r}_v| dudv$. The integral

$$\boxed{\iint_R dS = \iint_R |\vec{r}_u \times \vec{r}_v| \; dudv}$$

is the **surface area** of the surface.

Graph case

If $\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$, then $\vec{r}_u = \langle 1, 0, f_u(u,v)\rangle$ and $\vec{r}_v = \langle 0, 1, f_v(u,v)\rangle$. We get $\vec{r}_u \times \vec{r}_v = \langle -f_u, -f_v, 1\rangle$ and $|\vec{r}_u \times \vec{r}_v| = \sqrt{1 + f_u^2 + f_v^2}$.

Example: The surface area of a paraboloid $z = f(x,y)$ with $x^2 + y^2 \leq 1$ for example is $\iint_{u^2+v^2 \leq 1} \sqrt{1 + u^2 + v^2} \; dudv$ which in polar coordinates is $\int_0^{2\pi} \int_0^1 \sqrt{1 + r^2} \; r \; drd\theta$.

Sphere case

With $\vec{r}(\phi, \theta) = \langle \rho \sin(\phi)\cos(\theta), \rho \sin(\phi)\sin(\theta), \rho\cos(\phi)\rangle$ we parametrize a **sphere** of radius ρ. We have $|\vec{r}_\phi \times \vec{r}_\theta| = \rho^2 \sin(\phi)$ which explains also that if we go in to spherical coordinates we had an integration factor $\rho^2 \sin(\phi) d\phi \theta d\rho$. We can rewrite this as $dSd\rho$ and think of this as a thin spherical shell element of area dS and thickness $d\rho$.

Example: The sphere has surface area

$$\int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi) \; d\phi d\theta = 4\pi \; .$$

Planar case

With $\vec{r}(s,t) = \vec{P} + s\vec{v} + t\vec{w}$ we have $\vec{r}_s = \vec{v}$ and $\vec{r}_t = \vec{w}$ and $\vec{r}_s \times \vec{r}_t = \vec{v} \times \vec{w}$. If we look at the parameter domain $0 \leq s \leq 1, 0 \leq t \leq 1$, then we have the surface area of the **parallelogram** spanned by the two vectors $\vec{v}$ and $\vec{w}$.

Revolution case

In the **surface of revolution** case, we have $\vec{r}(\theta, z) = \langle g(z)\cos(\theta), g(z)\sin(\theta), z \rangle$. We compute $|\vec{r}_\theta \times \vec{r}_z| = g(z)\sqrt{1 + g'(z)^2}$.

Example: Let us take for example a trumpet with $g(z) = 1/z$ and assume that z ranges from 1 to b. The trumpet has now surface area

$$\int_0^{2\pi} \int_1^b \frac{1}{z}\sqrt{1 + \frac{1}{z^4}}\, dz \geq \int_0^{2\pi} \int_1^b \frac{1}{z}\, dz = 2\pi \log(b)\ .$$

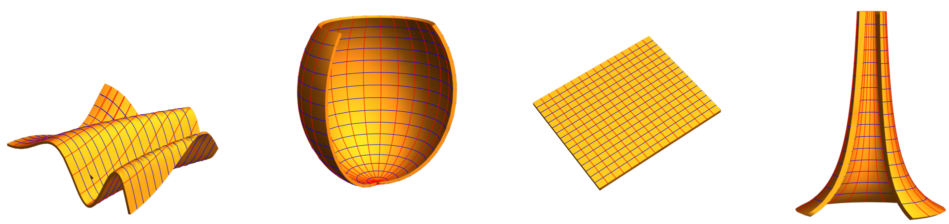

Figure 1. Graph, sphere, planar and revolution case.

17. Surface integral

If $f(u, v)$ is a **density function**, we can look at the **surface integral**

$$\boxed{\iint_R f\, dS = \iint_R f(u, v)|\vec{r}_u \times \vec{r}_v|\ dudv}$$

An important example is $f(u, v) = 1$, in which case we just have the surface area. It is important to think about the surface integral as a generalization of the surface area integral. Sometimes this can be a bit puzzling.

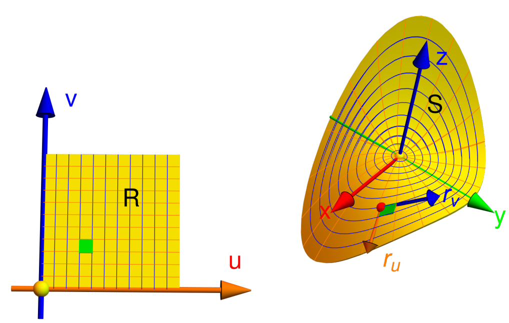

Figure 2. The surface area element $|\vec{r}_u \times \vec{r}_v|\ dudv$.

Example: Let us look for example at the case $f(\theta, z) = z^2$ and let S be the cone surface $\vec{r}(\theta, z) = \langle z\cos(\theta), z\sin(\theta), z \rangle$ with $|z| \leq 1$ for which we have $|\vec{r}_z \times r_\theta| = \sqrt{2}z$ and $\iint_S f\ dS = \int_0^{2\pi} \int_{-1}^1 \sqrt{2}z^3\ dz = 0$ which obviously is not right. The mistake was that $|\vec{r}_z \times r_\theta| = |z|$. We can rectify that by computing $2\int_0^{2\pi} \int_0^1 \sqrt{2}z^3\ dzd\theta$.

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