MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 19: Vector fields and Curl

Planar vector fields

A **planar vector field** is a **vector valued function** $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$. You can draw it by attaching a vector at every point (x,y).

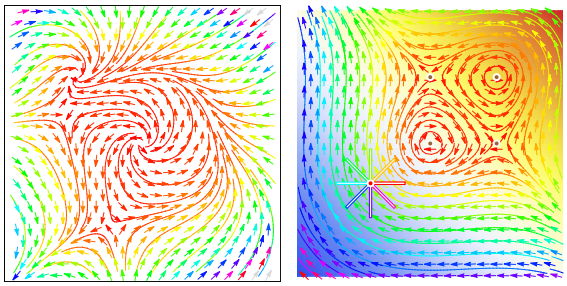

FIGURE 1. A vector field in the plane attaches to every point a vector The curl measures how fast a wheel placed into the field rotates counter clockwise.

Similarly as you learned how to match functions with graphs, we learn how to match vector field formulas with pictures of fields. Similarly as **partial derivatives** of a function gave information like slopes, partial derivatives like Q_x or P_y give valuable information about the field. We do not have slope but we have a notion of how the field "turns". This is called the curl.

The curl

The **curl** of a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is the scalar field $\boxed{\text{curl}(\vec{F}(x,y)) = Q_x - P_y}$. We also write $\text{curl}_2(\vec{F}(x,y))$.

Examples

Example 1. The field $\vec{F}(x,y) = \langle P, Q\rangle = \langle -y, x\rangle$ obviously rotates. We measure constant curl 2.

Example 2. The field $\vec{F}(x,y) = \langle x, y\rangle$ does not rotate. Indeed, we measure zero curl because $Q_x - P_y = 0$.

Example 3. A third example is $\vec{F}(x,y) = \langle y, 0\rangle$. Also this vector field has constant curl -1. A wheel placed in turns clockwise.

Example 4. The following example will be important next week when we compute area $\vec{F}(x,y) = \langle 0, x \rangle$. It is the simplest field which has constant curl 1.

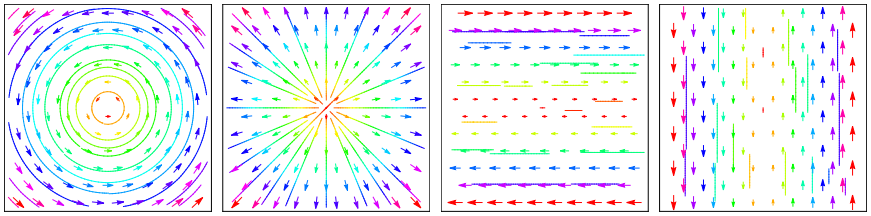

FIGURE 2. Examples 1 to 4.

SPACIAL VECTOR FIELDS

A **space vector field** is a vector valued function $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$. Here is an example

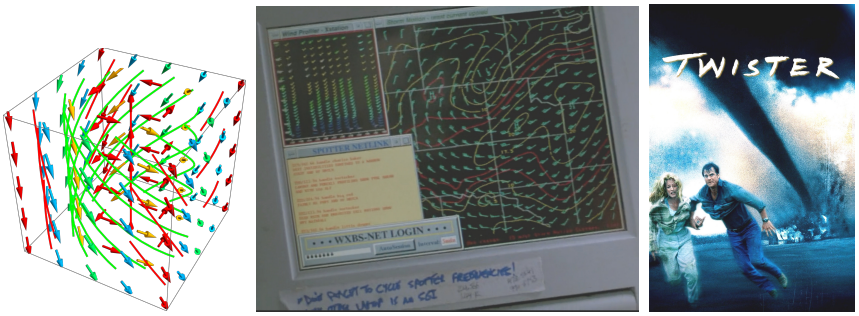

FIGURE 3. A vector field in space. Right: a screen shot from the movie "Twister, 1996" shows a vector field.

EXAMPLES

Example 1. This field models a tornado $\vec{F}(x,y,z) = \langle -y, x, 1 \rangle$. Welcome to the twister carrying you to the land of OZ

Example 2. The ABC vector field $\vec{F}(x,y,z) = \langle \sin(z)+\cos(y), \sin(x)+\cos(z), \sin(y)+\cos(x) \rangle$.

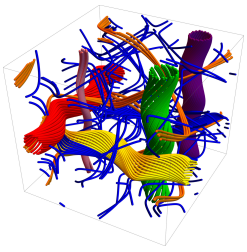

FIGURE 4. The Arnold-Beltrami-Childress (ABC) field appears in Chaos theory. It is an exact solution of the Euler's equation describing inviscid fluids.