MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 21: Green's Theorem

The theorem

Let G be a region in the plane bounded by a closed curve $\vec{r}(t) = \langle x(t), y(t) \rangle$. Given also $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ a vector field. Green's theorem relates the **line integral**

$$\int_C \vec{F}\ d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$$

with the double integral $\int\int_G \text{curl}(\vec{F})\ dA$ of the curl. Remember that $\text{curl}(\vec{F}) = Q_x - P_y$.

$$\boxed{\int_C \vec{F}\ d\vec{r} = \int\int_G \text{curl}(\vec{F})\ dA.}$$

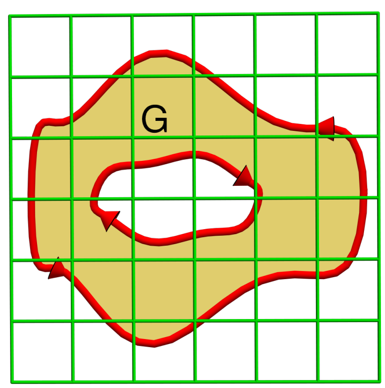


FIGURE 1. Green's theorem relates the integral of the curl of $\vec{F}$ over the region with the line integral of the field over the boundary.

Example

Let $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ wit $0 \leq t \leq 2\pi$ be a circle and $\vec{F} = \langle y, x^2 \rangle$ be a vector field. We have already computed $\int_0^{2\pi} \langle \sin(t), \cos(t)^2 \rangle \cdot \langle -\sin(t), \cos(t) \rangle\ dt = \int_0^{2\pi} -\sin^2(t) + \cos^3(t)\ dt = -\pi$. The vector field has curl $Q_x - P_y = 2x - 1$. Integrated over the disc gives also $-\pi$.

Area

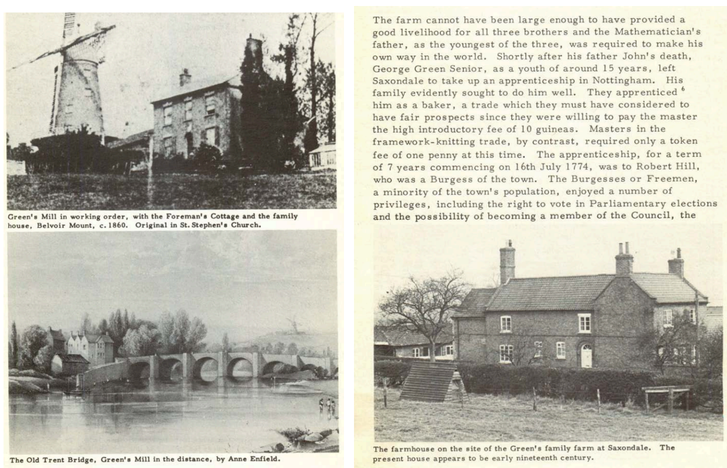
The field $\vec{F}(x, y = \langle 0, x\rangle$ has constant curl 1. If a region is bound by a curve $\vec{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, then Green's theorem assures that

$$\int_C \vec{F}\ \vec{dr} = \int_0^{2\pi} \langle 0, a\cos(t)\rangle \cdot \langle -a\sin(t), b\cos(t)\rangle\ dt = \int_0^{2\pi} ab\cos^2(t)\ dt = \pi ab\ .$$

We have computed the area of an ellipse of width a and height b.

History

George Green 1893-1841) was a British mathematician who first described a mathematical framework for electricity and magnetism paving the way for Clerk Maxwell and Lord Kelvin.



Green's Mill in working order, with the Foreman's Cottage and the family house, Belvoir Mount, c. 1860. Original in St. Stephen's Church.

The Old Trent Bridge, Green's Mill in the distance, by Anne Enfield.

The farm cannot have been large enough to have provided a good livelihood for all three brothers and the Mathematician's father, as the youngest of the three, was required to make his own way in the world. Shortly after his father John's death, George Green Senior, as a youth of around 15 years, left Saxondale to take up an apprenticeship in Nottingham. His family evidently sought to do him well. They apprenticed [6] him as a baker, a trade which they must have considered to have fair prospects since they were willing to pay the master the high introductory fee of 10 guineas. Masters in the framework-knitting trade, by contrast, required only a token fee of one penny at this time. The apprenticeship, for a term of 7 years commencing on 16th July 1774, was to Robert Hill, who was a Burgess of the town. The Burgesses or Freemen, a minority of the town's population, enjoyed a number of privileges, including the right to vote in Parliamentary elections and the possibility of becoming a member of the Council, the

The farmhouse on the site of the Green's family farm at Saxondale. The present house appears to be early nineteenth century.

Figure 2. From a biography about Green.

The planimeter

The **planimeter** is a device that can compute the area of regions using Green's theorem. The vector $F(x, y)$ is a unit vector perpendicular to the second leg $(a, b) \to (x, y)$ if $(0, 0) \to (a, b)$ is the second leg. Given (x, y) we find (a, b) by intersecting two circles. The magic is that the curl of F is constant 1.

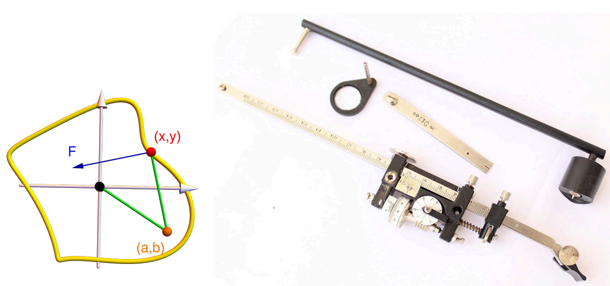


Figure 3. The planimeter can compute area.