MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 22: Path independence

From Conservative to Path independence

A vector field $\vec{F} = \langle P, Q \rangle$ is called **path independent** if for any two points A, B and any two paths C_1, C_2 connecting A with B, the line integrals agree $\int_{C_1} \vec{F} \cdot \vec{dr} = \int_{C_2} \vec{F} \cdot \vec{dr}$. A vector field $\vec{F}$ has the **closed loop property** or is called **conservative** if for any closed path C the line integral is zero $\int_C \vec{F} \cdot \vec{dr} = 0$.

Conservative implies path independence.

Proof: Two paths C_1, C_2 from A to B define a closed loop C which can be written as $C_1 - C_2$, meaning C_2 is traversed backwards. The integral along C is zero implies that the line integrals along C_1 and C_2 agree.

From Path independence to Gradient

If $\vec{f}$ is a function, then $\vec{F} = \nabla f = \langle f_x, f_y \rangle$ is called the **gradient** of f.

Path independence means gradient field.

Proof: We can **construct** a function $f(x,y) = \int_{C(x,y)} \vec{F} \cdot \vec{dr}$, where $C(x,y)$ is a path from $(0,0)$ to (x,y). The function is independent of the path. To check $f_x = P$, chose a path going from $(0,0)$ to $(0,y)$ to (x,y), then check $f_x(x,y) = P(x,y)$ at this point. To verify $f_y = Q$, chose a path going from $(0,0)$ to $(x,0)$ and from there to (x,y). Now check $f_y(x,y) = Q(x,y)$.

From Gradient field to zero curl

As usual, we assume that $\vec{F}$ is defined everywhere in the plane. A vector field is called **irrotational** if $\text{curl}(F) = 0$ everywhere in the plane.

Gradient fields are irrotational.

Proof: If $\langle P, Q \rangle = \langle f_x, f_y \rangle$, then $Q_x - P_y = f_{yx} = f_{xy} = 0$ by **Clairaut's theorem**.

From Zero curl to conservative

Irrotational implicates conservative.

Proof. Given a closed loop C bounding a region G. **Green's theorem** implies $\int_C \vec{F} \cdot \vec{dr} = \iint_G \text{curl}(\vec{F}) \, dA = 0$.

Summary

We have verified with a "Merry go round proof" the following theorem:

> If $\vec{F}$ is defined everywhere, then four things are equivalent: 1) $\vec{F}$ is path independent 2) $\vec{F}$ is conservative 3) $\vec{F}$ is a gradient field 4) $\vec{F}$ is irrotational.

If $\vec{F}$ is not defined everywhere, the first two things are still equivalent but 3) and 4) can fail. An example is the **vortex** $\vec{F} = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$, which is defined everywhere except at $(0,0)$. One can compute $Q_x - P_y = 0$ everywhere except at $(0,0)$. One can also check that $\vec{F} = \nabla f$ with $f(x,y) = \arctan(y/x)$ but which has a problem that it is discontinuous. There is no global gradient on $\mathbb{R}^2 \setminus \{(0,0)\}$. The fancy way to tell this is that is a "the punctured plane has a **non-trivial cohomology**". Just that you have something to bluff with at you next cocktail party ...

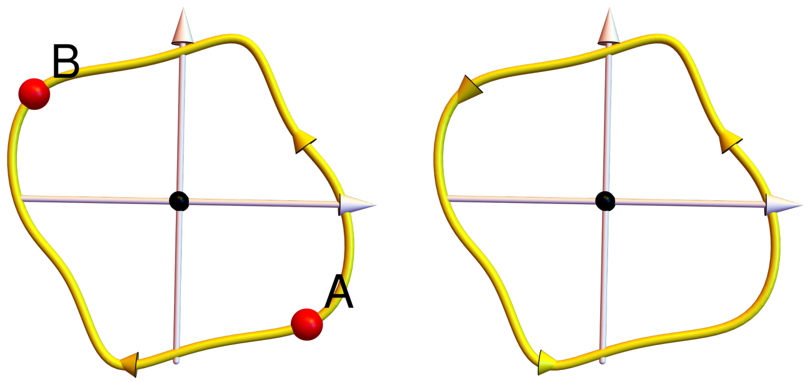

Figure 1. Path independence and conservative are equivalent.

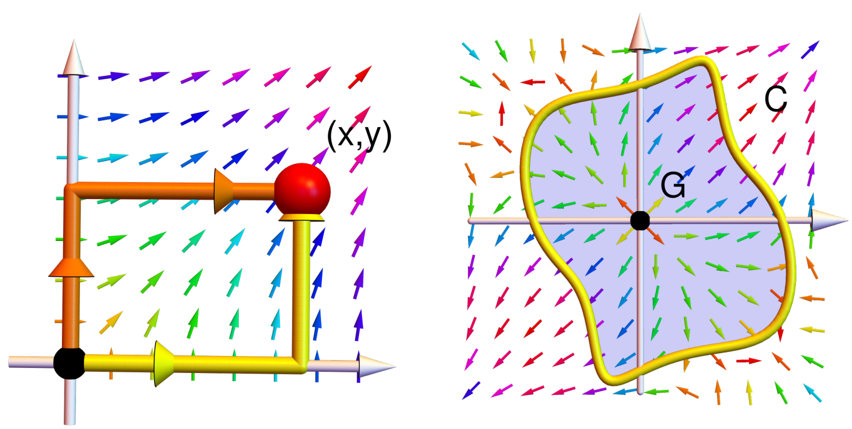

Figure 2. Path independence and gradient field are equivalent: the potential f can be constructed as a line integral from $(0,0)$ to (x,y). Irrotational implies conservative by Green's theorem.