MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 23: Linear Approximation

Square root magic

Suppose you want to compute the square root $\sqrt{x} = \sqrt{67}$ without computer. Calculus can help. We know that $y_0 = \sqrt{x_0} = \sqrt{64} = 8$. Now, we also know the slope of $f(x) = \sqrt{x}$ at x_0 as $m = 1/(2\sqrt{x_0}) = 1/16$. If we replace the graph of $f(x)$ by the line $L(x) = y_0 + m(x - x_0)$ we can estimate $\sqrt{67}$ by $L(67) = 8 + 3/16 = 9.1875$. This is quite close to the actual value $\sqrt{67} = 9.18535....$

Linearization

The function

$$\boxed{L(x) = f(x_0) + f'(x_0)(x - x_0)}$$

is called the **linearization** of f at x_0. The graph of L is tangent to the graph of f at x_0.

In two dimensions, the linearization is done both for x and y. Given a function $f(x, y)$ of two variables and a point (x_0, y_0), define the linearization as

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The graph of L is tangent to the graph of f at (x_0, y_0).

The gradient

With the **gradient notation** $\nabla f(x, y) = \langle f_x, f_y \rangle$ we can rewrite this as

$$\boxed{L(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle}$$

we see that this looks exactly like the linearization in one dimensions.
Given a function $f(x, y)$ we get a vector field $\vec{F}(x, y) = \langle f_x, f_y \rangle$ called the **gradient vector field**. We have seen last time that it is irrotational.

Chain rule

The **chain rule** in one dimensions is $\frac{d}{dt} f(g(t)) = f'(g(t))g'(t)$. For example $\frac{d}{dt} \sin(t^5) = \cos(t^5)5t^4$.

A function $f(x, y)$ evaluated on a curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ satisfies $\frac{d}{dt} f(\vec{r}(t)) = f_x(x(t), y(t))x'(t) + f_y(x_y, y(t))y'(t)$ This can be written more elegantly using the gradient notation

$$\boxed{\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}$$

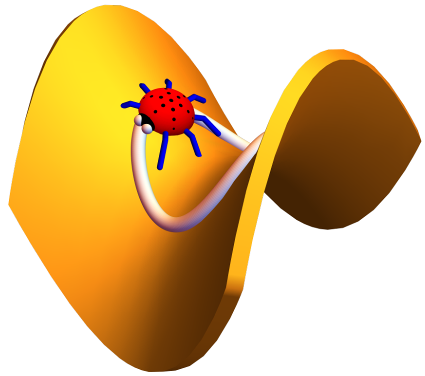

FIGURE 1. A bug moves along a circle $\vec{r}(t)$ near a surface of $z = f(x, y)$. The rate of change $d/dt f(\vec{r}(t))$ is the gain of height the bug climbs in unit time. It depends on the velocity $\vec{r}\,'(t)$ and the gradient $\nabla f(\vec{r}(t))$. If the speed were constant 1, then $d/dt f(\vec{r}(t))$ is the slope. It is the scalar component onto the gradient vector.

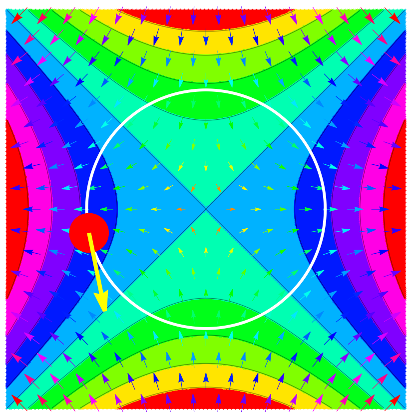

FIGURE 2. Here we see the bug picture in the contour picture. The gradient field $\nabla f(x, y) = \langle f_x, f_y \rangle$ is a vector field in the plane. The path $\vec{r}(t) = \langle x(t), y(t) \rangle$ is a path in the plane.

Preview to line integrals

Here we can turn back to line integral. Remember that $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ is the line integral. The chain rule allows us to rewrite this as

$$\int_a^b \frac{d}{dt} f(\vec{r}(t))\ dt\ .$$

Do you see how this can be simplified? We will come to this later.

Section Oliver Knill, knill@math.harvard.edu, Harvard College, Fall 2022