OLIVER KNILL, MATH 21A

Lecture 27: Flux integrals II

Flux reminder

$$\int_S \vec{F} \cdot d\vec{S} = \int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dudv$$

is the **flux** of $\vec{F}$ through the surface S parametrized by $\vec{r} : R \to S$. Here is an intuitive example: if $\vec{F}$ is a constant vector field and S is a closed surface, then the flux of $\vec{F}$ through S is zero. An important special case is if $\vec{F} \cdot \vec{n} = C$ is constant, where $\vec{n}$ is the **unit normal vector**. **Then** $\int_S \vec{F} \cdot d\vec{S} = C \int_S dS$ **is** C **times the surface area of** S**.** For example, if S is the unit sphere, then $\int\int_S \langle x,y,z \rangle \cdot d\vec{S}$ can be computed by noticing $\vec{n} = \langle x,y,z \rangle$ so that $\vec{F} \cdot \vec{n} = 1$. The result is 4π.

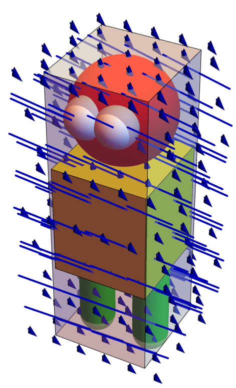

Figure 1. The soaked student problem.

The soaking student problem

Suppose you go rush to class in the morning during a rainy day. The flux of the rain $\vec{F} = \langle 0,0,-1 \rangle$ through a surface the amount of water soaked up from the rain. You are a cuboid of dimensions $2,2,6$ and go from home $A = (0,0,0)$ and the classroom is at $B = (10,0,0)$ with a speed v. How can you minimize the water you catch? Go fast or slow?

If you walk with speed v, you have to walk for $100/v$ seconds. In your coordinate system, where you are at rest, the rain is the vector field $\vec{F} = \langle v,0,-1 \rangle$. Two surfaces are exposed to the rain. Your front side A with normal vector $\vec{a} = \langle -1, 0, 0 \rangle$ has area 12 Your top side B has normal vector $\langle 0, 0, 1 \rangle$ has area 4. The flux through your front shirt A is $12\vec{F} \cdot \vec{a} = 12v$. The flux through your hair is $\vec{F} \cdot \vec{b}4 = 4$ and is not affected on how fast your go. The total flux you catch is $4 + 12v$. You walk for $10/v$ seconds. Do you now see whether it is better to walk fast (catching a lot of rain through the front) or slow getting soaked from above but for longer?

The solar panel problem

Oliver likes to brag about his moon property of the size of Manhatten. One of Oliver's Moon base units will contain a half sphere dome $x^2 + y^2 + z^2 \leq 100$, $z \geq 0$. Assume the cosmic radiation vector field is $\vec{F} = \langle 1, 0, -1 \rangle$. Is there more flux of cosmic radiation going through the top part of the dome or through the base of the dome? Parametrize both problems.

1. Bottom: We parametrize the floor as $\vec{r}(u, v) = \langle u, v0 \rangle$ with $\vec{r}_u \times \vec{r}_v = \langle 0, 0, 1 \rangle$ with $R = \{u^2 + v^2 \leq 1\}$. The flux is $\iint_R \vec{F} \cdot \vec{r}_u \times \vec{r}_v \, dudv = -\pi$.
2. Dome: We parametrize as $\vec{r}(u, v) = \langle \sin(v)\cos(u), \sin(v)\sin(u), \cos(v) \rangle$ with $\vec{r}_u \times \vec{r}_v = \sin(v)\langle \sin(v)\cos(u), \sin(v)\sin(u), \cos(v) \rangle$. We have $\vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v = \sin^2 v \cos(u) - \sin(v)\cos(v)$. The flux is now

$$\int_0^{2\pi} \int_0^{\pi/2} \sin^2 v \cos(u) - \sin(v)\cos(v) \, dvdu$$

which is $-\pi$.
We see that the flux through the bottom is the same than the flux through the top. We will see an other explanation later.

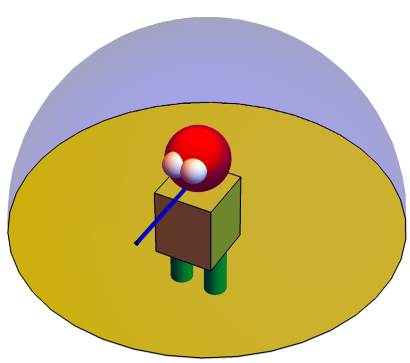

Figure 2. Oliver in his moon base, ready to defend it with a blow gun. This is my land!