# Lecture 28: Maxima and minima

## One dimensions

Pièrre Fermat realized that if $f(x)$ is a function of one variable like $f(x) = 3x^3 - x$ and $x_0$ is a maximum or minimum, then $f'(x_0) = 0$. The reason is that if $f'(x_0) \neq 0$, then we could increase $f$ with $x = x_0 + tf'(x_0)$ for some small $t$. So, critical points, points where $f'(x) = 0$ are candidates for maxima or minima. It does not guarantee a maximum or minimum as the example $f(x) = x^3$ shows. But if $f'(x_0) = 0$ and $f''(x_0) > 0$, then $x_0$ is a minimum and if $f'(x_0) = 0$ and $f''(x_0) < 0$, then $x_0$ is a maximum. This **second derivative test** immediately follows from **quadratic approximation**: at a critical point which is

$$Q(x) = f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 = Z + \frac{A}{2}u^2 \ .$$

We see that the sign of $A = f''(x_0)$ determines whether we have a maximum or minimum. When finding maxima and minima, we also have to take into account the boundary. Local maxima and minima which are critical points away from the boundary. The largest maximum or minimum overall on a domain is called a **global maximum** or **global minimum**.

## Critical points

For a function $f(x, y)$ of two variables, a point $(x_0, y_0)$ is called a **critical point**, if $\nabla f(x_0, y_0) = 0$. Critical points are candidates for maxima and minima again because of Fermat's principle: if $\nabla f(x_0, y_) \neq \langle 0, 0\rangle$, then we can increase $f$ by looking at the point $(x_0, y_0) + t\nabla f(x_0, y_0)$ for small $t > 0$ and get a larger value. A critical point does not have to be a maximum or minimum. Now, there is even the example of saddle points like $f(x, y) = x^2 - y^2$, where $(0, 0)$ is a critical point but where we obviously do not have a maximum, nor a minimum. When doing quadratic approximation, we have seen three typical situations.

1. $f(x, y) = x^2 + y^2$. This is a **local minimum** at $(x_0, y_0) = (0, 0)$ because $\nabla f(x, y) = \langle 2x, 2y\rangle$ and $f_{xx} = 2$ and $D = f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$.

2. $f(x, y) = -x^2 - y^2$. This is a **local maximum** at $(x_0, y_0) = (0, 0)$ because $\nabla f(x, y) = \langle -2x, -2y\rangle$ and $f_{xx} = -2$ and $D = f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$.

3. $f(x,y) = x^2 - y^2$. This is a **saddle point** at $(x_0, y_0) = (0,0)$ because $\nabla f(x,y) = \langle 2x, -2y \rangle$ $D = f_{xx} f_{yy} - f_{xy}^2 = -4 < 0$.

4.$f(x,y) = x^3 - 3xy^2$ is an example with $D = 0$ because $\nabla f(x,y) = \langle 3x^2 - 3y^2, -6xy \rangle$ and $D = f_{xx} f_{yy} - f_{xy}^2 = -36x^2 - 36y^2$ which is zero at $(0,0)$.

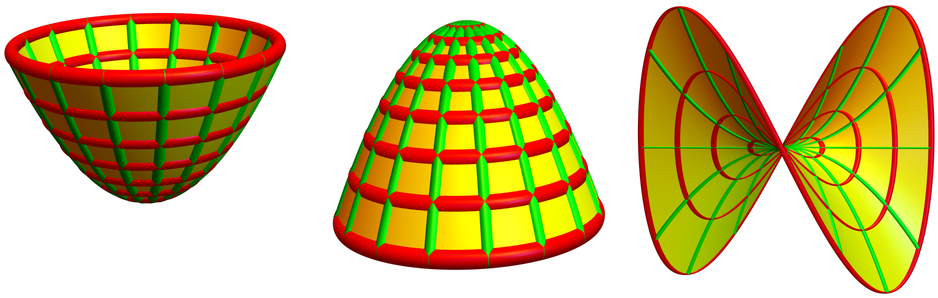

FIGURE 1. Minimum, Maximum and Saddle point

## SECOND DERIVATIVE TEST

The **discriminant** $D = f_{xx} f_{yy} - f_{xy}^2$ and $f_{xx}$ allow us to see cases, where know to have a local maximum, a local minim or a saddle point.

Assume $(x_0, y_0)$ is a **critical point** for $f(x,y)$.
If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ then $(x_0, y_0)$ is a **local minimum**.
If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ then $(x_0, y_0)$ is a **local maximum**.
If $D < 0$ then $(x_0, y_0)$ is a **saddle point**.
If $D = 0$, we need **other means** to decide.

**Proof:** This **second derivative test** follows from **quadratic approximation** $Q(x,y) =$

$$f(x_0, y_0) + \frac{f_{xx}(x_0, y_0)(x - x_0)^2 + 2 f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2}{2} .$$

A completion of square trick allows to write this as $Q(x,y) =$

$$Z + \frac{Au^2 + 2Buv + Cv^2}{2} = Z + \frac{A}{2}(u + \frac{Bv}{A})^2 + \frac{AC - B^2}{2A} v^2 .$$

We see now that if $D = CA - B^2$ and $A$ are positive, we have a minimum value $Z$ and that if $D$ and $A$ are negative, we have a maximal value $Z$ and that if $D < 0$, we have a saddle point at $(x_0, y_0)$, independently whether $A$ was positive or negative. If $A = 0$ and $C$ is not, we can switch coordinates $x \to y$ to get expressions with $A$ and $C$ switched. If $A = C = 0$, we have $Q = Z + Buv$ which gives a saddle.

## EXAMPLE

The function $f(x,y) = x^3/3 - x - (y^3/3 - y)$ has the gradient $\nabla f(x,y) = [x^2 - 1, -y^2 + 1]$. and so the critical points $(1,1)$,$(-1,1)$,$(1,-1)$ and $(-1,-1)$. We see $f_{xx} = 2x$ and $D = -4xy$. For $(1,1)$ and $(-1,-1$ we have $D = -4$ and so a saddle points. For $(-1,1)$ we have $D = 4$, $f_{xx} = -2$ and so a local maximum For $(1,-1)$ we have $D = 4$, $f_{xx} = 2$ and so a local minimum.