MULTIVARIABLE CALCULUS

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Lecture 29: Lagrange

Lagrange equations

If a function $f(x,y)$ is maximized on a curve $g(x,y) = c$, we want the gradients of f and g to be parallel. The reason is that if $\vec{r}(t)$ parametrizes the curve, then $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$ implies that $\nabla f(\vec{r}(t))$ is perpendicular to $\vec{r}\,'(t)$ and so parallel to the normal vector $\nabla g(\vec{r}(t))$. We therefore have to solve the system:

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$g(x,y) = c$$

for the unknowns x, y, λ, where the constant λ is called the **Lagrange multiplier**.

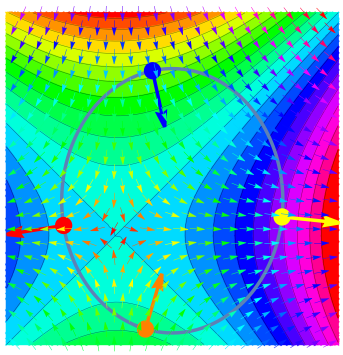

Figure 1. The **Theorem of Lagrange** tells that if (x_0, y_0) is a maximum or minimum of f under the constraint $g = c$, then either the Lagrange equations hold or then $\nabla g(x_0, y_0) = \langle 0, 0 \rangle$.

Example

Problem: Use the Lagrange method to solve the problem to minimize $f(x,y) = x^2 + y^2 + xy$ under the constraint $g(x,y) = 3x + 4y = 26$.

Solution: The Lagrange equations are

$$\begin{aligned} 2x + y &= \lambda 3 \\ x + 2y &= \lambda 4 \\ 3x + 4y &= 26 \end{aligned}$$

By cross multiplication we get rid of the constant λ $(2x+y)4 = 3(x+2y)$. $3x+4y = 26$. We can subtract the two equations to solve for x and get $x = 2$. From the third equation we get then $y = 5$.

Example

Here is an example of a minimum, without the Lagrange equations being satisfied:
Problem: Use the Lagrange method to solve the problem to minimize $f(x, y) = x$ under the constraint $g(x, y) = y^2 - x^3 = 0$.
Solution: The Lagrange equations are

$$\begin{aligned} 1 &= \lambda \cdot (-3x^2) \\ 0 &= \lambda 2y \\ y^2 - x^3 &= 0 \end{aligned}$$

From the second equation we see that either $\lambda = 0$ or $y = 0$ but $\lambda = 0$ does not work with the first equation. So $y = 0$. From the third equation we get $x = 0$. But now, with $x = 0$ we have a problem with the first equation. The Lagrange equations have no solution! But there is obviously a minimum $x = 0, y = 0$. This is a point, where $\nabla g(x, y) = \langle 0, 0\rangle$. Examples like that force the theorem to include the case, where $\nabla g = 0$. Indeed, if $\nabla g = 0$, then still ∇g is perpendicular to ∇f. It is just that with our choice $\nabla f = \lambda \nabla g$, we did not allow $\nabla g = 0$. We would have to write $\nabla g = \lambda \nabla f$.

A reduction to a problem without constraints

Economists sometimes rewrite the Lagrange equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0$ as $\nabla F(x, y, \lambda) = \langle 0, 0, 0\rangle$, where $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$. The third component of the gradient of F gives the constraint $g(x, y) = 0$. This reduces the Lagrange problem to an extremization problem for a function F with one variable more. But you see the limitations of this also with the previous example, where $F(x, y, \lambda) = x - \lambda(y^2 - x^3)$. There are critical points of F, even so the constraint extremization problem has a solution.

More variables

The Lagrange method works for arbitrary many variables.
Problem: maximize the entropy of a **tetrahedral dice**. The probability to throw 1 is a, the probability to throw 2 is b the probability to throw 3 is c and the probability to throw 4 is d. The entropy of the dice is

$$f(a, b, c, d) = -a\log(a) - b\log(b) - c\log(c) - d\log(d) \ ,$$

where $\log = \ln$ is the natural log. Because we deal with a probability distribution, we have the constraint $g(a, b, c, d) = a + b + c + d = 1$.
Solution: the Lagrange equations are

$$f_a = \lambda g_a, f_b = \lambda g_b, f_c = \lambda g_c, f_d = \lambda g_d, g(a, b, c, d) = 1 \ .$$

This means $-1 - \log(a) = -1 - \log(b) = -1 - \log(c) = -1 - \log(d) = \lambda$ from which we can deduce that $a = b = c = d = 1/4$ is the maximum. The **fair dice** is the dice with maximal Shannon entropy.