MULTIVARIABLE CALCULUS

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Lecture 30: Global extrema

Global maxima

If G is a region with boundary C and $f(x,y)$ is a function of two variables, we can look at local maxima and minima in the interior by looking at critical points $\nabla f(x,y) = \langle 0,0 \rangle$. On the boundary C look for maxima and minima using the **Lagrange method**. If $f(x_0,y_0)$ is larger or equal than any other $f(x,y)$ in G, then (x_0,y_0) is called a **global maximum**. **Globla minima** are defined in the same way.

Extremal value theorem

A region is called **closed** if it contains all boundary points. It is called **bounded** if it is contained in some large ball $\{x^2+y^2 \leq r\}$. A closed and bounded region is also called **compact**. Here is **extremal value theorem** first proven by Bolzano:

> **Theorem:** **Theorem:** If G is a closed and bounded region and f is continuous, then f has a global maximum on G.

The supremum of all values $f(x)$ is a finite number M. Otherwise, there would exist an infinite sequence x_k of numbers in G such that $f(x_k) \to \infty$. But as there exists a convergent sub-sequence $y_l = x_{k_l}$ converging to a point y, we would have $f(y_k) \to f(y) = \infty$ which contradicts continuity of f. Let x_k be a sequence so that $f(x_k) \to M$. Now x_k has an accumulation point x because G is bounded and closed. Since $f(x) = M$ and $f(z) \leq f(y)$ for all other z this is a global max. We need only continuity for this theorem and not the existence of derivatives. If we can differentiate, then we can look for maxima using calculus.

FIGURE 1. Bernard Bolzano (1781-1848) brought some rigor into analysis.

Examples

To find the global maximum of f on a closed bounded domain, make first a list of all the local maxima in the interior possibly using the second derivative test, then make a list of all the extrema on the boundary, possibly using Lagrange. The maximum of this combined list is the global maximum.

Problem: Find the global maxima and minima of $f(x,y) = x^4 + y^4$ on the disk $x^2 + y^2 \leq 1$. **Solution:** For the interior, we see $\nabla f(x,y) = \langle 0,0 \rangle$ at $(0,0)$. The second derivative test is inconclusive because $D = 0$. But we see that $(0,0)$ is a global minimum as $f \geq 0$ everywhere on G and $f(0,0) = 0$. We look with Lagrange for constrained maxima and minima: $4x^3 = \lambda 2x, 4y^3 = \lambda 2y, x^2 + y^2 = 1$ shows that either $x = 0, y = \pm 1$ or $y = 0, x = \pm 1$ or then $x = \pm y = \pm 1/\sqrt{2}$. The points on the axis are global maxima.

Problem: Find the global extrema $f(x,y) = 1 - \sqrt{x^2 + y^2}$ on $x^2 + y^2 \leq 1$. **Solution:** The gradient of f is singular at $(0,0)$. But we see that $f(0,0) = 1$ is larger or equal than any other value to that $(0,0)$ is the global maximum. Also clear is that $f(x,y) \geq 0$ on the domain and that $f(x,y) = 0$ on the boundary. Every boundary point is a global minimum.

Problem: Decide whether $f(x,y) = x^3 y - x^4(y-1) + xy$ has a global maximum or minimum. **Solution:** If we look at $y = 1$. The function is then $f(x,1) = x^3 + x$ which has no global maximum nor a global minimum. $f(x,y)$ inherits that.

The island theorem

Here is a fun fact. Assume we have an island in the form of a bounded region G. The height function $f(x,y)$ of the island is equal to 0 at the boundary and every critical point has non-zero D. The **island theorem** tells that the number of local maxima plus the number of local minima minus the number of saddle points is equal to 1.

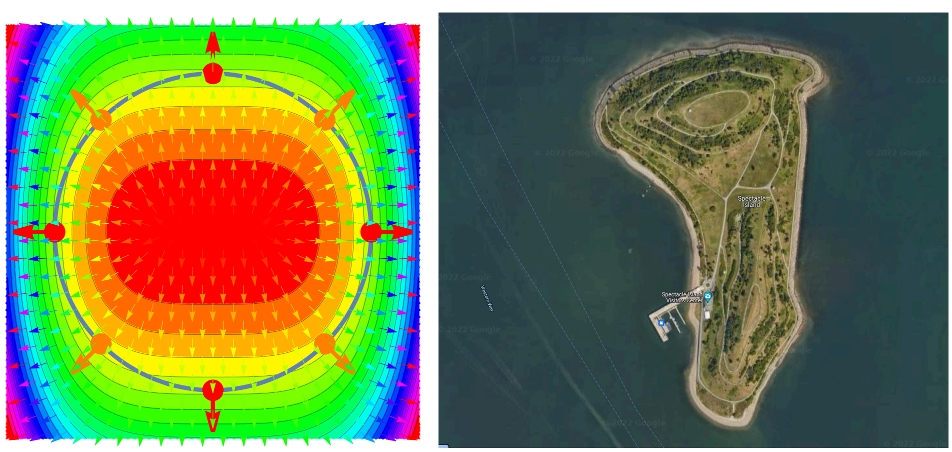

Figure 2. To the left we see the example with $f(x,y) = x^4 + y^4$. Spectacle island in Boston has two local hills and a saddle points.