

MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 31: More extrema

STATISTICS

Data fitting is the problem to minimize the sum of the squares of the data points to a line $y = mx + b$. Assume we want to fit the data $(3, 0)$, $(2, 3)$, $(-1, 2)$, $(1, 2)$, $(0, 1)$ best. We want to minimize the function

$$f(x, y) = (b + 3m - 0)^2 + (b + 2m - 3)^2 + (b - m - 2)^2 + (b + m - 2)^2 + (b - 0m - 1)^2.$$

There is one critical point $(-1/5, 9/5)$. The best line is $y = 9/5 - x/5$.

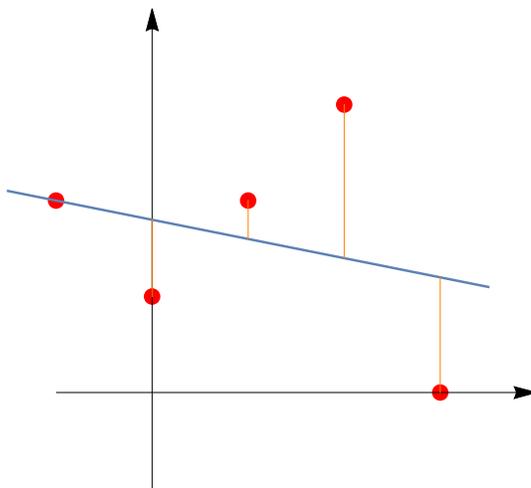


FIGURE 1. A Data fitting problem

This can be useful for building models. A famous data fitting problem was solved by Cobb and Douglas in the context of **economics**.

MECHANICS

We have already seen the **double pendulum** for which the potential energy $-\cos(x) - \cos(y)$ had four equilibrium points: a maximum, a minimum and two saddle points.

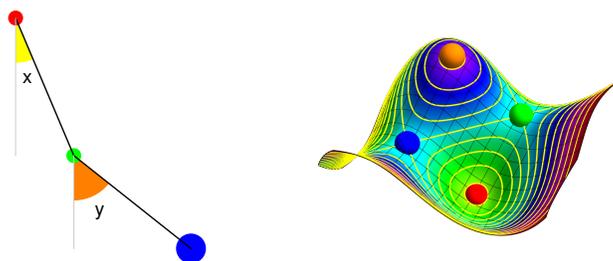


FIGURE 2. A double pendulum. The distance function to the origin has a maximum, a minimum and two saddle points. To the right we see a polynomial function $f(x, y)$ which has this type of critical points.

PHYSICS

Snells law in optics explains how light moves in media like water. One calls the effect **refraction**. One of the laws is the reflection law telling that incoming and outgoing angle are the same. This is a Lagrange problem. If $g(x, y) = c$ is a curve and the refraction index in one part is a and the index in the other b , then the light going from a point A to a point B has to minimize $f(x, y) = ad(A, (x, y)) + bd((x, y), B)$. For air the refraction index is close to $a = 1$ for water it is close to $b = 4/3$. The angles of incidence i and the angle of refraction r are related by **Snell's law** $a \sin(i) = b \sin(r)$. The light ray tries to travel for less time in medium with higher refraction index.

GEOMETRY

The problem of **maximizing the area** f of a polygon of fixed length g is a Lagrange problem for the variables $(x_1, x_1), \dots, (x_n, y_n)$, the points of the polygon. Using Greens theorem with the vector field $\vec{F} = \langle -y, x \rangle / 2$, one can see that $f(x_1, \dots, y_n) = \sum_{k=1}^n (x_k y_{k+1} - y_k x_{k+1}) / 2$ is the area of the polygon. The length to constrain is $g(x_1, \dots, y_n) = \sum_{k=1}^n \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2} = L$. By looking at three adjacent points P, Q, R in the polygon, one can see that the distances $d(P, Q)$ and $d(Q, R)$ have to be the same. The maximal area is therefore a regular polygon.

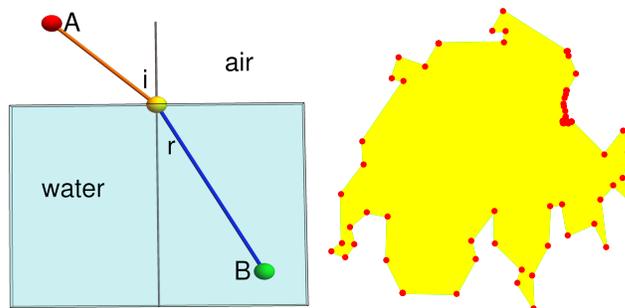


FIGURE 3. Snell's law is a Lagrange problem. The area of a polygon can be computed with Green.