

MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 32: Grad-Curl-Div

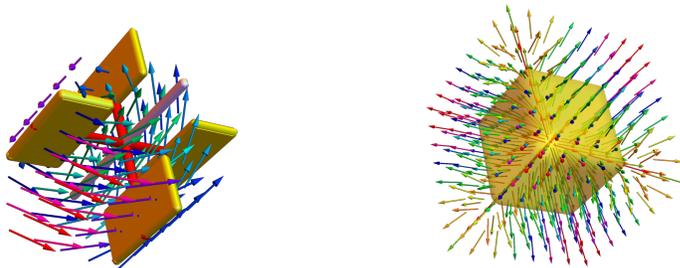


FIGURE 1. The curl $\text{curl}(\vec{F})$ at a point P gives the rotation axes and intensity of a field at P . The divergence $\text{div}(\vec{F})$ at a point P measures the expansion rate of the field at P . The justification of these pictures will be given with Stokes theorem and the divergence theorem.

CALCULUS IN ONE DIMENSION

In one dimensions, there is one **derivative** f' and one **integral**. The two are linked by the **fundamental theorem of calculus** $\int_a^b f'(x) dx = f(b) - f(a)$.

CALCULUS IN TWO DIMENSIONS

In two dimensions, we have two type of derivatives $\text{grad}(f)$ and $\text{curl}(F)$. We have also two type of integrals: the **line integral** $\int_a^b \vec{F} \cdot d\vec{r}$ and the **double integral** $\iint_R f(x, y) dA$. The gradient and line integral pair up to the **fundamental theorem of line integrals**, the curl and the double integral pair up to **Green's theorem**.

CALCULUS IN THREE DIMENSIONS

In three dimensions we have three type of derivatives $\text{grad}(f)$ and $\text{curl}(F)$ and $\text{div}(F)$. We also have three type of integrals: the **line integral** $\int_a^b \vec{F} \cdot d\vec{r}$ the **flux integral** $\iint_S \vec{F} \cdot d\vec{S}$ and the **triple integral** $\iiint_R f(x, y, z) dV$. The gradient and line integral pair up to the **fundamental theorem of line integrals**, the curl and the flux will

pair up to **Stokes theorem** and the divergence and triple integral will pair up with the **divergence theorem**.

THE DEFINITION OF THE CURL

The two dimensional curl had been associated with rotation in the plane. A rotation in the plane is determined by one angle only. The three dimensional curl is a notion that is associated with rotations in space. Rotations in space have three parameters. The definition of the curl of a field $\vec{F} = \langle P, Q, R \rangle$ is

$$\text{curl}(\vec{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle .$$

HOW TO REMEMBER IT

You see in the third component the curl $Q_x - P_y$ you are already familiar with. You can also see $\text{curl}(\vec{F}) = \nabla \times \vec{F}$, where $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$ is an differential operator. For example to compute the curl of $\langle -y, x, xyz \rangle$, write the nabla vector above it, then compute. We get $\langle xz, -yz, 2 \rangle$.

WHAT DOES IT MEAN?

This will only be clear when having seen Stokes theorem but one can see the vector $\text{curl}(\vec{F})$ at a point as having the direction in which one has to hold a paddle wheel so that it rotates fastest in the counter clockwise direction, when we hold the wheel into that direction. The speed of the rotation is the length of $\text{curl}(\vec{F})$.

THE DEFINITION OF THE DIVERGENCE

The divergence of a vector field field $\vec{F} = \langle P, Q, R \rangle$ is defined as

$$\text{div}(\vec{F}) = P_x + Q_y + R_z .$$

The divergence measures expansion or contraction of the field at some point.

HOW TO REMEMBER IT

Again, with the vector ∇ we can write $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$. This is not really a dot product but just a formal way to write $\partial_x P + \partial_y Q + \partial_z R$.

WHAT DOES IT MEAN?

If you look at the example $F = \langle x, y, z \rangle$, you compute $\text{div}(\vec{F}) = 3$. We see that this field expands. Place a box into space and let it flow with the field, then it will expand the box. The vector field $\langle x, 0, 0 \rangle$ expands in the x direction, the vector field $\langle 0, y, 0 \rangle$ expands in the y direction and the vector field $\langle 0, 0, z \rangle$ expands in the z direction.

IDENTITIES

There are two important identities: $\text{curl}(\text{grad}(f)) = \vec{0}$ and $\text{div}(\text{curl}(\vec{F})) = 0$. In words, this means that gradient fields are **irrotational** and that curl vector fields are **incompressible**. This terminology comes from hydro dynamics. A fluid which does not rotate is irrotational. A fluid that can not be compressed is called incompressible.