

# MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

## Lecture 33: Stokes Theorem

### THE THEOREM

Assume  $S$  is a surface with boundary  $C$  oriented so that the surface is to the “left” if “up” is in the surface normal  $\vec{n}$  and “forward” is the velocity  $\vec{r}'$ . **Stokes theorem:**

$$\text{Theorem: } \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

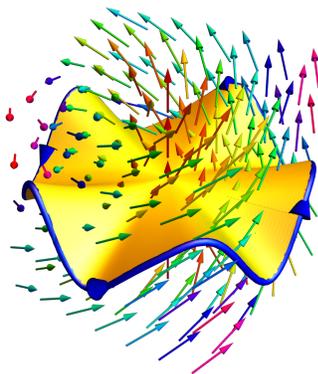


FIGURE 1. Stokes theorem.

Written out, if  $\vec{r} : R \rightarrow S$  is the parametrization of  $S$  and  $\vec{r} : I \rightarrow C$  is the parametrization of  $C$ , then

$$\iint_R \text{curl}(\vec{F})(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv = \int_I \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt .$$

The theorem involves cross product, dot product, vector fields, parametrized curves, parametrized surfaces and is therefore a culmination of topics we have been working on.

The intuition is similar as for Green’s theorem: do the computation for a small parallelogram  $\vec{r}_u du \times \vec{r}_v dv$  on the surface. Adding the fluxes through all these surface elements, gives the total flux. Summing up the line integrals has the effect that the interior line integrals cancel and only the line integral along the boundary  $C$  remains.

## EXAMPLES

1) Find the flux of the curl of the vector field  $\langle -y + z \sin(x^2), x + zy^3, (xyz)^3 \rangle$  through the upper half sphere  $x^2 + y^2 + z^2 = 1, z \geq 0$  oriented upwards.

**Solution:** Computing the curl integrating its flux would be quite difficult. We use Stokes theorem. The boundary  $C$  is the curve  $\vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ . We have  $\vec{F}(\vec{r}(t)) = \langle -\sin(t), \cos(t), 0 \rangle$  and  $\vec{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$ . The result is  $2\pi$ .

2) Find the line integral  $\int_C \vec{F} \cdot d\vec{r}$ , where  $C : \vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$  and where  $\vec{F} = \langle x^3 - y, y^3 + x, z^5 \rangle$ . Use a convenient surface  $S$  which has  $C$  as a boundary.

**Solution:** Take the disk of radius 1 in the  $xy$ -plane as the surface. It is parametrized by  $\vec{r}(x, y) = \langle x, y, 0 \rangle$  and has normal vector  $\vec{r}_u \times \vec{r}_v = \langle 0, 0, 1 \rangle$ . The curl of  $\vec{F}$  is  $\langle 0, 0, 2 \rangle$ . The flux is  $\iint_R 2 \, dudv = 2\pi$ .

## REMARKS

A) Stokes is Greens theorem transported into three dimensions: in the special case  $\vec{F} = \langle P, Q, 0 \rangle$  where  $\text{curl}(\vec{F}) = \langle 0, 0, Q_x - P_y \rangle$ , and where  $\vec{r}(u, v) = \langle u, v, 0 \rangle$  then  $\text{curl}\vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v = Q_x - P_y$  and  $\iint_R Q_x - P_y \, dA = \int_C \vec{F} \cdot \vec{r}$ .

B) Assume that  $\vec{F} = \text{grad}(f) = \langle f_x, f_y, f_z \rangle$  is a gradient field. Then  $\text{curl}(\vec{F}) = \langle 0, 0, 0 \rangle$ . The flux of the curl of the vector field through the surface  $S$  is zero. Because the boundary of the surface consists of closed curves, we get by the fundamental theorem of line integrals also zero for the line integral.

## APPLICATIONS

Stokes theorem is useful in fluid dynamics and electrodynamics.

One of the Maxwell equations for electric  $\vec{E}$  and magnetic  $\vec{B}$  fields.  $\frac{d\vec{B}}{dt} = -\text{curl}(\vec{E})$  motivates to look at the flux of  $\vec{B}$  through a surface  $S$ . Stokes theorem renders it a line integral of  $\vec{E}$  which is electric potential.

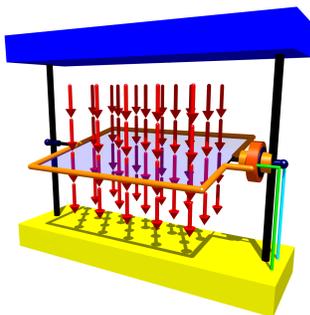


FIGURE 2. The flux of the magnetic field  $\vec{B}$  through a surface is by Stokes theorem the line integral of  $\vec{E}$  through  $C$ .