

MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 34: Divergence Theorem

THE THEOREM

Let \vec{F} be a vector field and E a solid which has as a boundary a surface S oriented so that the normal vectors point outwards. The divergence theorem is:

$$\textbf{Theorem: } \iiint_E \operatorname{div}(\vec{F}) dV = \iint_S \vec{F} \cdot d\vec{S}$$

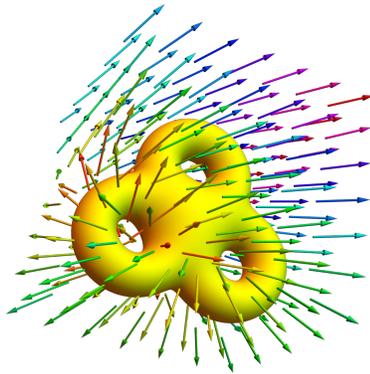


FIGURE 1. The divergence theorem.

Written out, if $\vec{r} : R \rightarrow S$ is the parametrization of S , then

$$\iiint_E \operatorname{div}(\vec{F})(x, y, z) dV = \iint_R \vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v \, dudv .$$

Together with Stokes theorem, the divergence theorem involves all topics we have been working on. The divergence theorem is the only integral theorem in three dimensions which involves triple integrals. The proof is done by proving it for cubes and fields like $\vec{F} = \langle P, 0, 0 \rangle$ first, then add things up in general. The intuition is that $\operatorname{div}(\vec{F})(x, y, z)$ measures how much field is “created” at a point (x, y, z) . The theorem becomes then a conservation law. It tells that whatever has been created or removed in the solid D will have to balance with the total flux that is coming in or leaving the solid.

EXAMPLES

1) Find the flux of the vector field $\vec{F} = \langle x - y + z \sin(y^2), z + 3y + zx^3, 5z + (xy)^4 \rangle$ through the sphere $x^2 + y^2 + (z - 1)^2 = 49$ oriented inwards.

Solution: The divergence of \vec{F} is 9. The divergence theorem therefore tells that the flux through the surface oriented outwards is 9 times the volume of the ball included by the surface. This is $9(4\pi 7^3/3)$. But since we want to compute flux when the orientation is inward, the result is $-9(4\pi 7^3/3)$.

2) Use $\vec{F}(x, y, z) = \langle x, y, z \rangle$ to compute the volume of the unit sphere.

Solution: The divergence of the field is 3. The vector field is perpendicular to the boundary surface S and has length 1 on the surface. The flux of the field through S therefore the surface area 4π of the sphere. The divergence theorem tells us $\iiint_E 3 dV = 4\pi$. The volume of the sphere therefore is $4\pi/3$.

REMARKS

1) Assume that $\vec{F} = \text{curl}(\vec{A})$ is the curl of an other vector field \vec{A} . Then $\text{div}(\vec{F}) = 0$. Integrating this over the solid E is zero. But since S is closed, also the flux of $\text{curl}(\vec{A})$ through S is zero. We have seen this also as a consequence of Stokes theorem.

2) There is also a 2 dimensional version of the divergence theorem $\iint_R P_x + Q_y dA = \int_C (\vec{G} \cdot \vec{n}) |\vec{r}'(t)| dt$. But after rotating $\vec{G} = \langle P, Q \rangle$ to $\vec{F} = \langle -Q, P \rangle$ and rotating the velocity vector $\langle x', y' \rangle$ and scaling it to get the normal vector $\vec{n} = \langle -y', x' \rangle / |\vec{r}'|$ we see $\text{div}G = \text{curl}(F)$ and $(\vec{G} \cdot \vec{n}) |\vec{r}'(t)| = \vec{F} \cdot \vec{r}'$ so that in two dimensions, the divergence theorem is just Green's theorem in disguise.

APPLICATIONS

1) A nice application of the divergence theorem is that it allows to **compute the volume of solids** E by integrating over the boundary. A computer would for example compute the flux of $\langle 0, 0, z \rangle$ through the boundary of a solid and get the volume. Making a Riemann sum over a two dimensional boundary is much less costly than summing over a three dimensional grid.

2) The divergence theorem is useful to **describe gravity**: Gauss realized that we can deduce Newton's law from the assumption that the gravitational field \vec{F} satisfies

$$\text{div}(\vec{F}) = -4\pi G\sigma .$$

where σ is the mass density and G the gravitational constant. This implies Newton's law $|\vec{F}| = GM/r^2$ but it is more general. We can for example compute the gravitational field inside the earth.

3) One of the **Maxwell equations** for the magnetic \vec{B} field is $\text{div}(\vec{B}) = 0$. This means that there are no magnetic monopoles. Given any closed surface S , the flux of the magnetic field through S is zero.