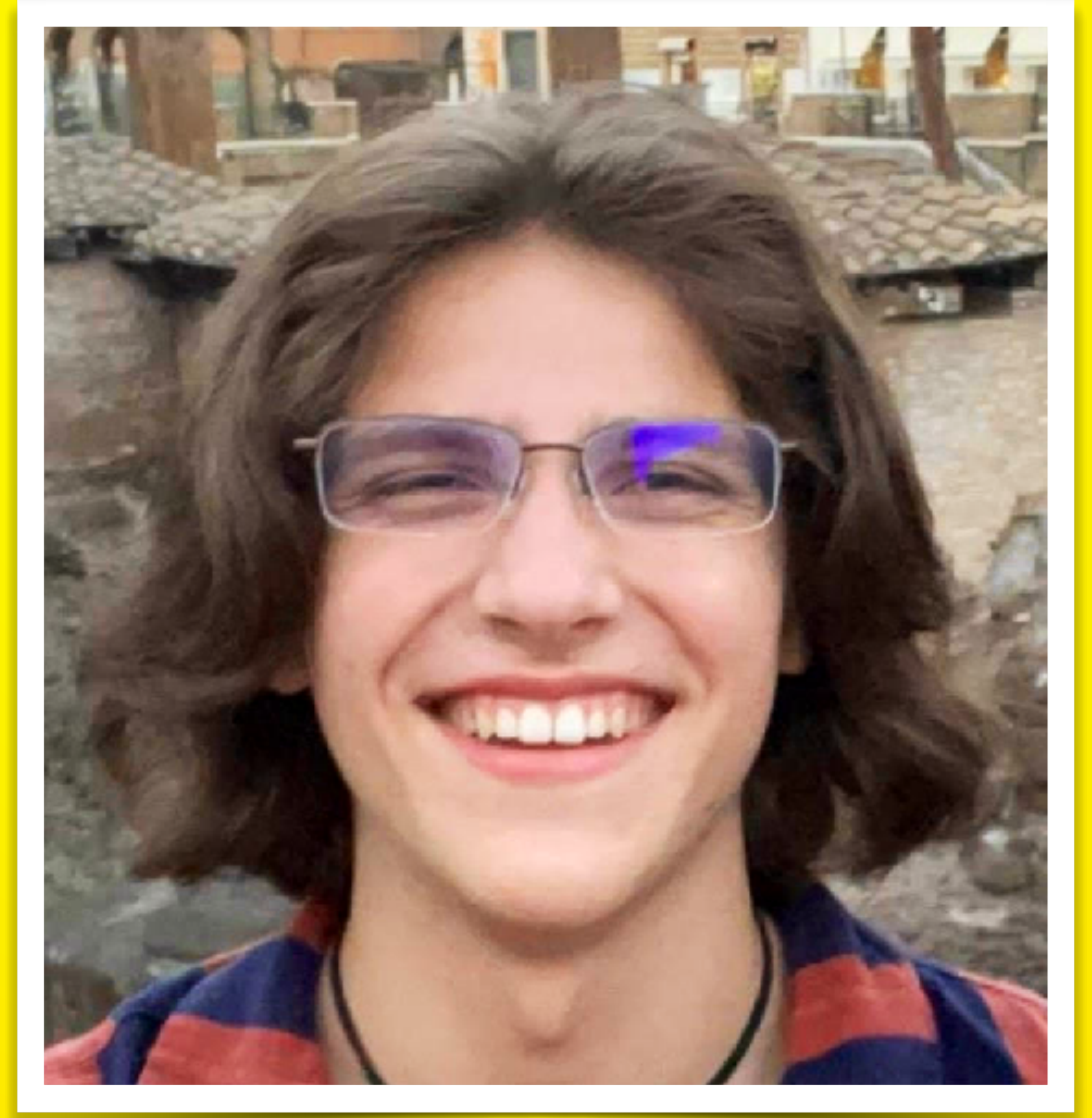
Lecture 1

Welcome to Math
21a, Fall 2022
Section **Oliver**
knill@math.harvard.edu
SC 432

CAs:
Marco Hansel,
Carel Zhang
Tali Wong

About our CAs

Tali Wong



Marco Hansel

Carol Zhang

Table of Contents

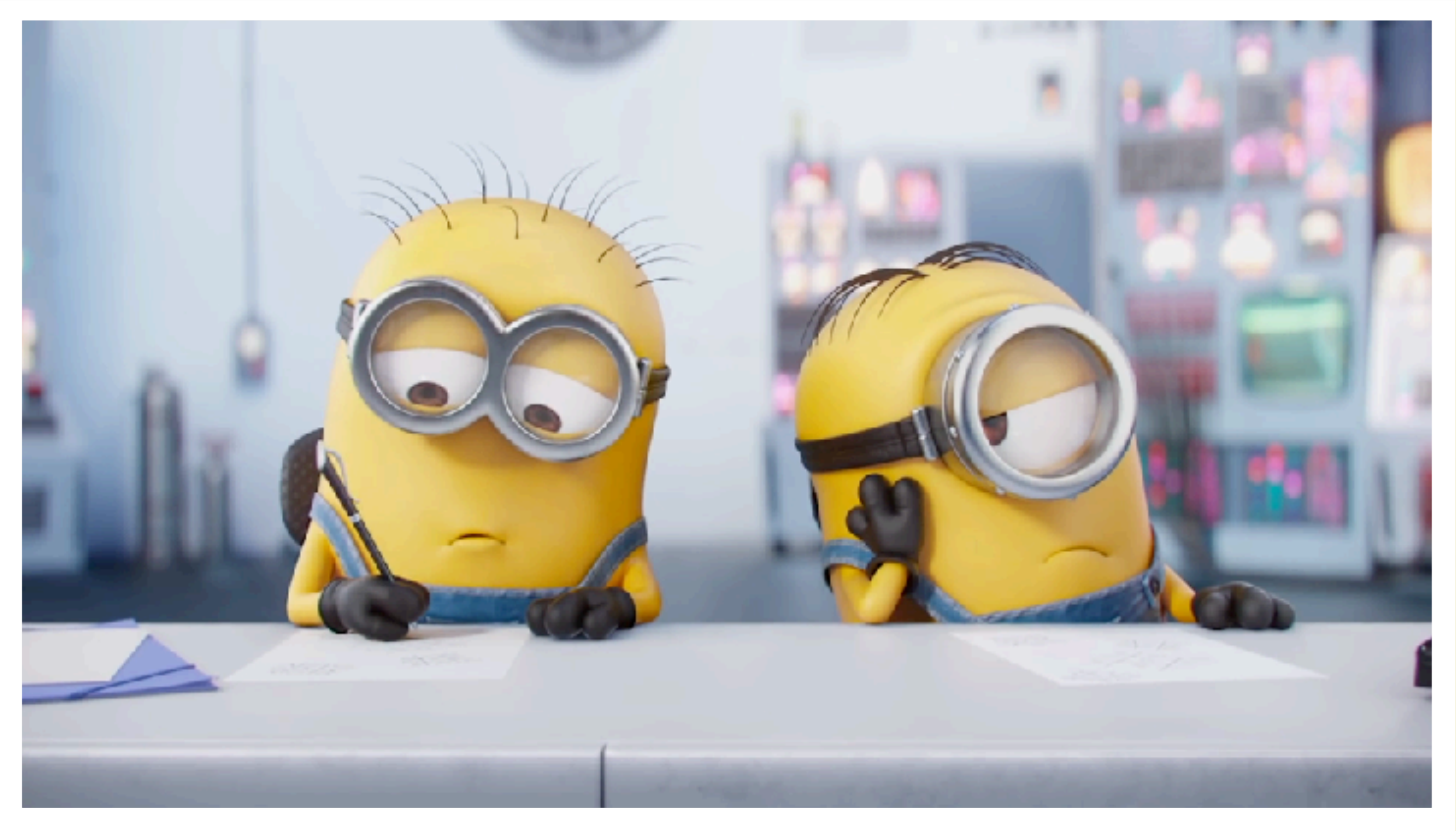
Part I

Introduction

About myself

Design:

I'm Oliver, was designed
in a secret science
lab in Switzerland



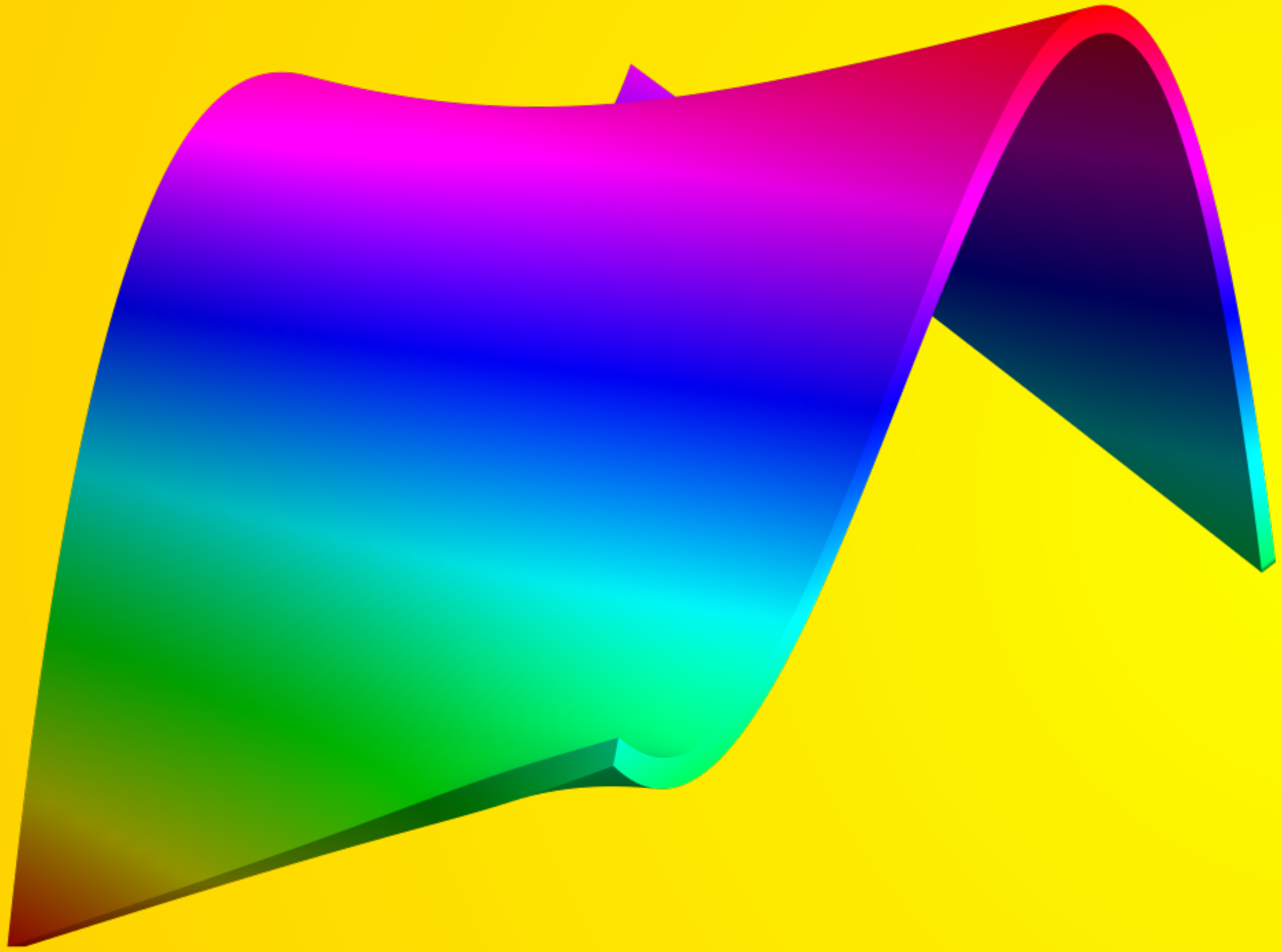
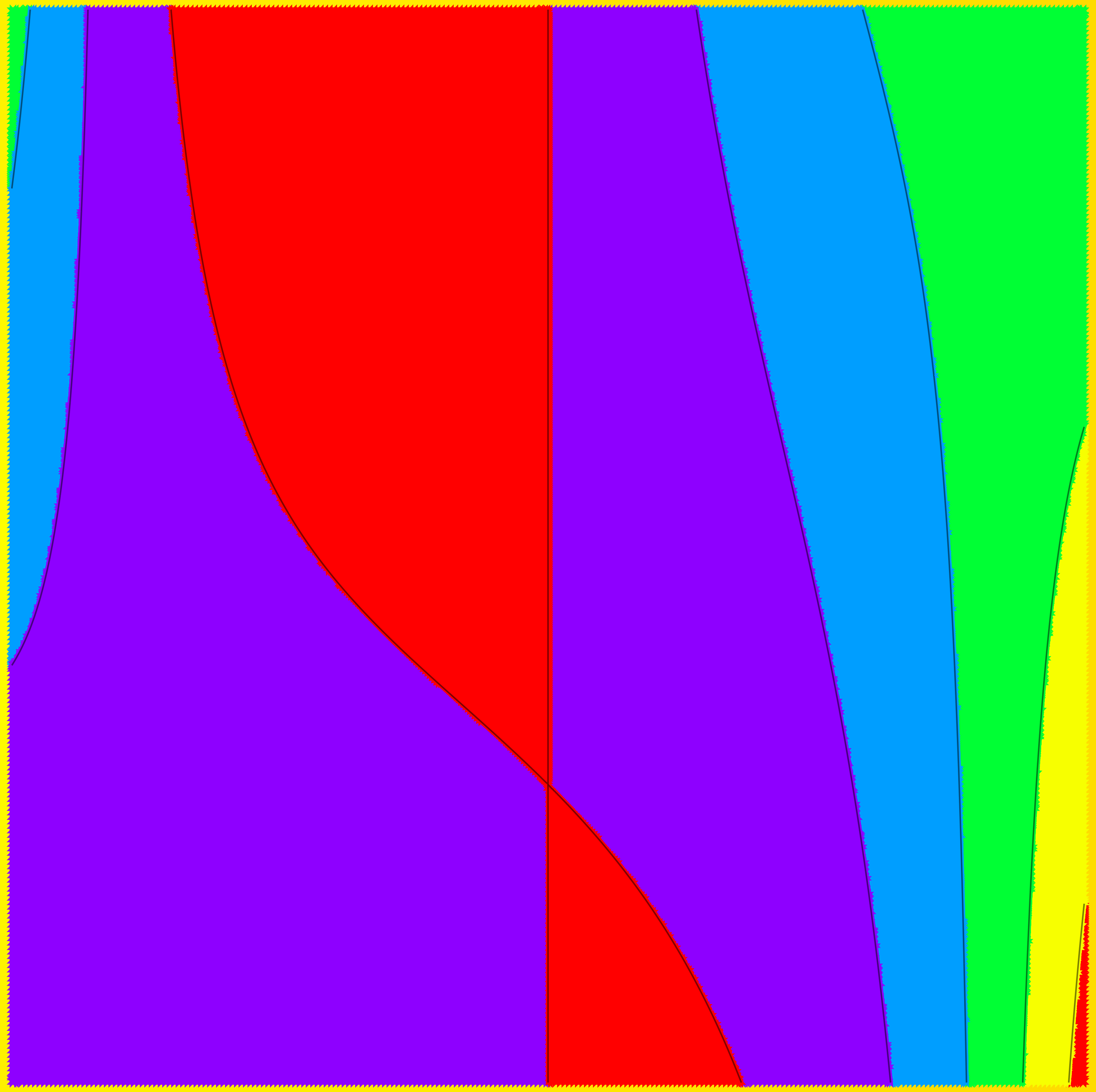
You can call
me Oliver

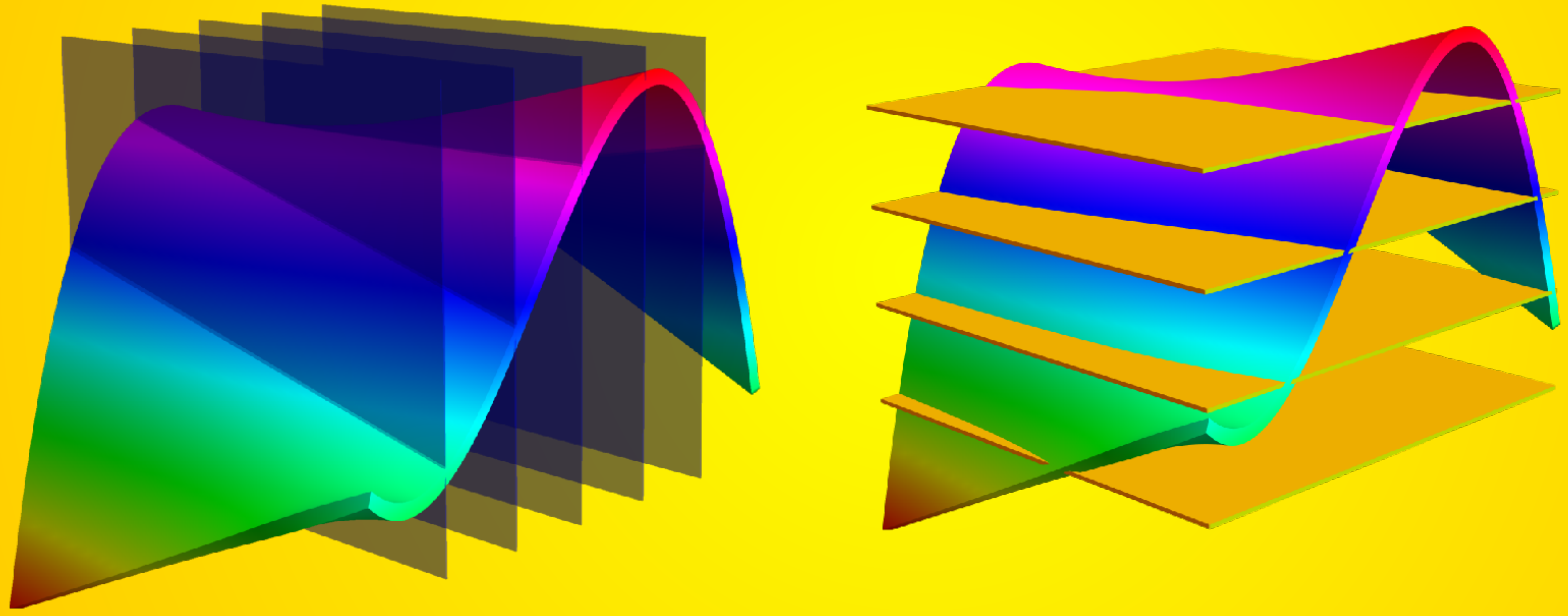
knill@math.harvard.edu


Läbe. Liebä.
Lachä.
TUI
Erläbe.
Gumpä. Flügä.
Ihtauchä.

Part 2

Graphs and Contours

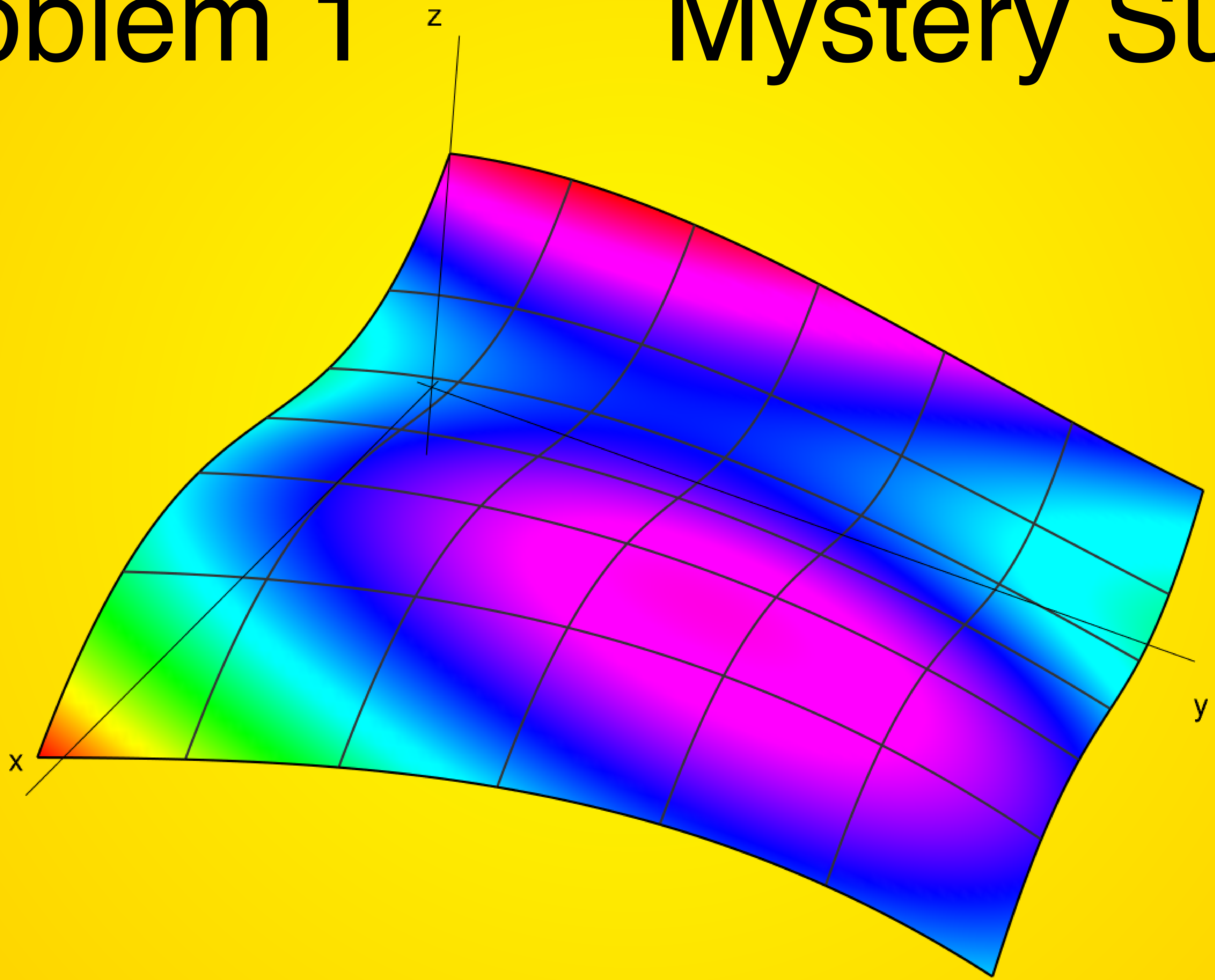


Part 3

Worksheet Part 1

(*) Worksheets are in general too long to be completed all here! We might do only one sometimes!

Problem 1 Mystery Surface



Problem 1

By the way

How to reconstruct the functi
f from the contours?

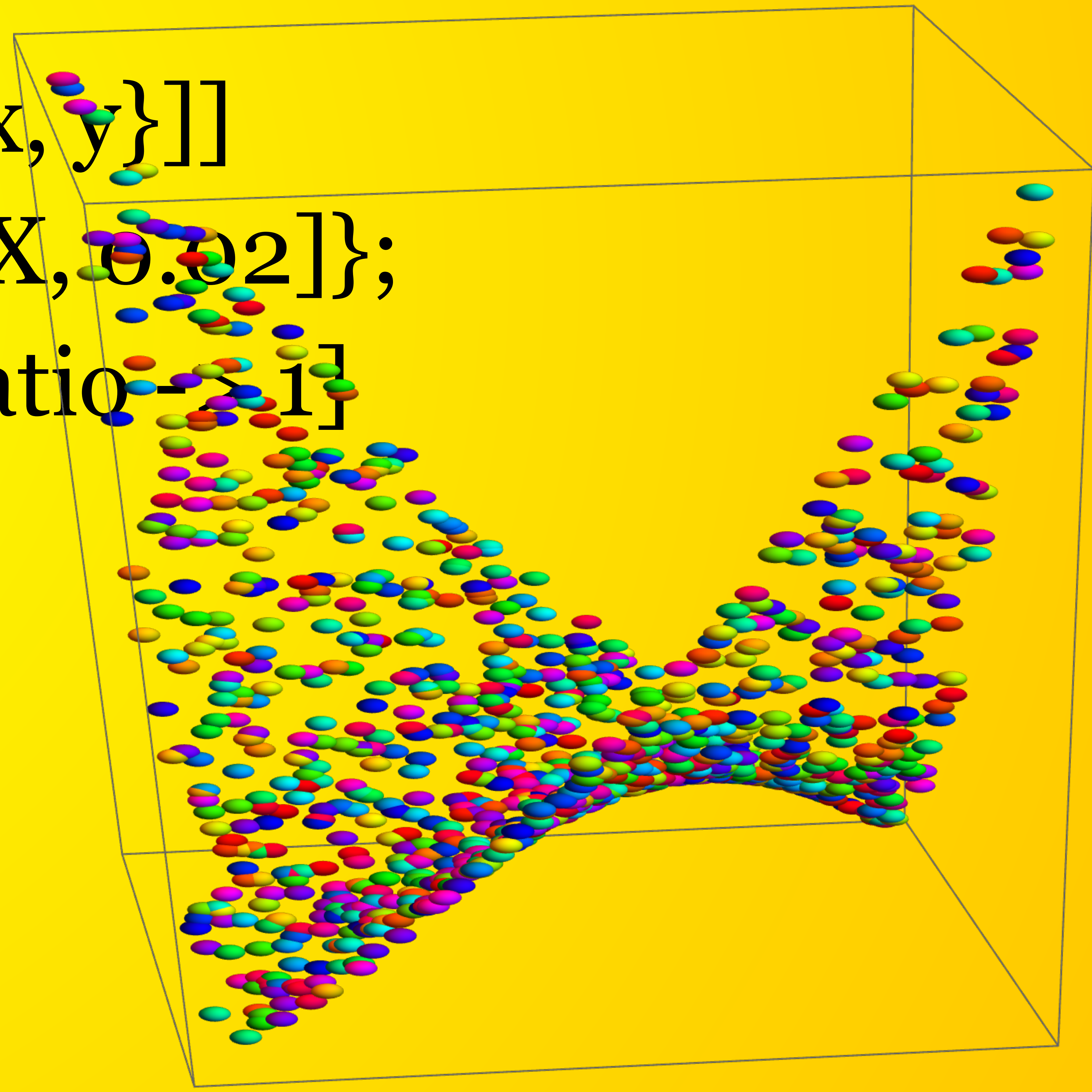
```
yy=0;  A00={{0.2 ,yy,3  },{0.65,yy,2.5},{1.4,yy,2.0},{5.1,yy,1.5},{5.65,yy,1.0}};
yy=0.5;A05={{0.3 ,yy,3  },{0.8 ,yy,2.5},{1.4,yy,2.0},{5.5,yy,1.5},{5.9, yy,1.0}};
yy=1.0;A10={{0.38,yy,3  },{0.95,yy,2.5},{5.3,yy,2.0},{5.5,yy,1.5},{5.8, yy,1.0}};
yy=1.5;A15={{0.4 ,yy,3  },{1.1 ,yy,2.5},{2.8,yy,2.5},{4.9,yy,2.5},{5.6, yy,2.0}};
yy=2.0;A20={{0.4 ,yy,3  },{1.2 ,yy,2.5},{2.5,yy,2.5},{5.3,yy,2.5},{5.8, yy,2.0}};
yy=2.5;A25={{0.35,yy,3  },{1.15,yy,2.5},{2.3,yy,2.5},{4.3,yy,3.0},{5.6, yy,2.5}};
yy=3.0;A30={{0.25,yy,3  },{0.95,yy,2.5},{2.3,yy,2.5},{3.3,yy,3.0},{4.95,yy,3.0},{5.7 ,yy,2.5}};
yy=3.5;A35={{0.25,yy,3  },{0.95,yy,2.5},{2.3,yy,2.5},{3.3,yy,3.0},{4.95,yy,3.0},{5.7 ,yy,2.5}};
yy=4.0;A40={{-0.1,yy,3  },{0.40,yy,2.5},{2.7,yy,2.5},{3.8,yy,3.0},{5.20,yy,3.0},{5.8 ,yy,2.5}};
yy=4.5;A45={{0.15,yy,3  },{1.0 ,yy,2.5},{2.7,yy,2.5},{4.1,yy,3.0},{5.20,yy,3.0},{5.9 ,yy,2.5}};
yy=5.0;A50={           {0.4 ,yy,2.0},{2.3,yy,2.0},{2.9,yy,2.5},{4.10,yy,3.0},{5.2 ,yy,3.0}};
yy=5.5;A55={           {0   ,yy,2.0},{2.7,yy,2.0},{3.6,yy,2.5},{5.80,yy,2.5}};
yy=6.0;A60={           {0.1 ,yy,2.5},{2.9,yy,2.5},{3.2,yy,2.0},{4.00,yy,2.5},{5.6 ,yy,2.4}};

f = Fit[Union[A00, A05, A10, A15, A20, A25, A30, A35, A40, A45, A50,
   A55, A60], {1, x, y, x^2, y^2, x*y, x*y^2, x^2*y, x^2*y^2, x^3, y^3, x^3 y, x y^3}, {x, y}]
```

Data fitting

```
R:=Random[]-0.5; data = Table[u = R;v=R;
  {u, v, u^2 + 2 v^2 - 5 u*v}, {1000}];
Chop[Fit[data, {1, x, x^2, y, y^2, x*y}, {x, y}]]
point[X_] := {Hue[Random[]], Sphere[X, 0.02]};
Graphics3D[Map[point, data], AspectRatio -> 1]
```

1. x^2 - 5. x y + 2. y^2

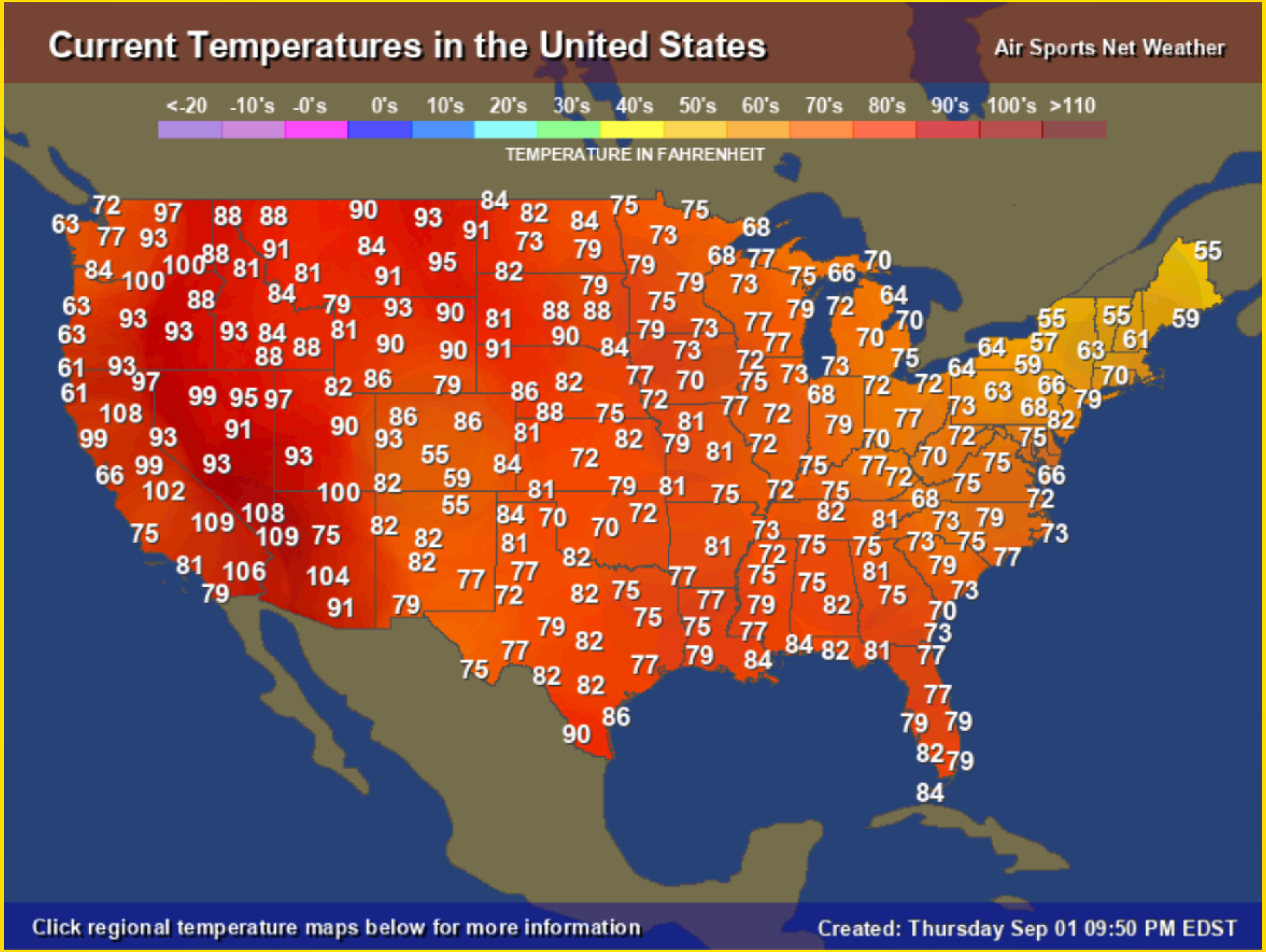



Cool!

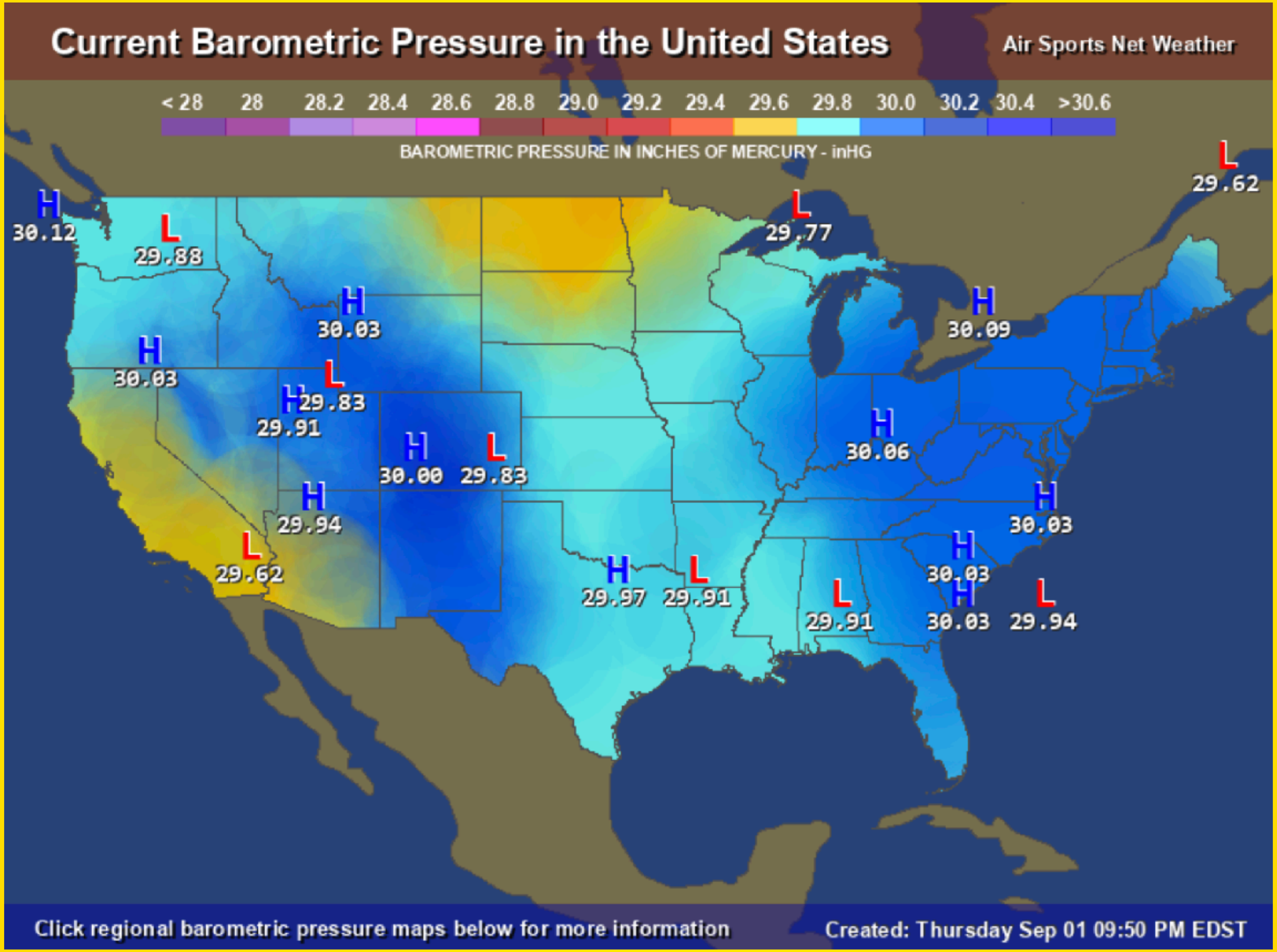
Part 4

Applications

Temperature map

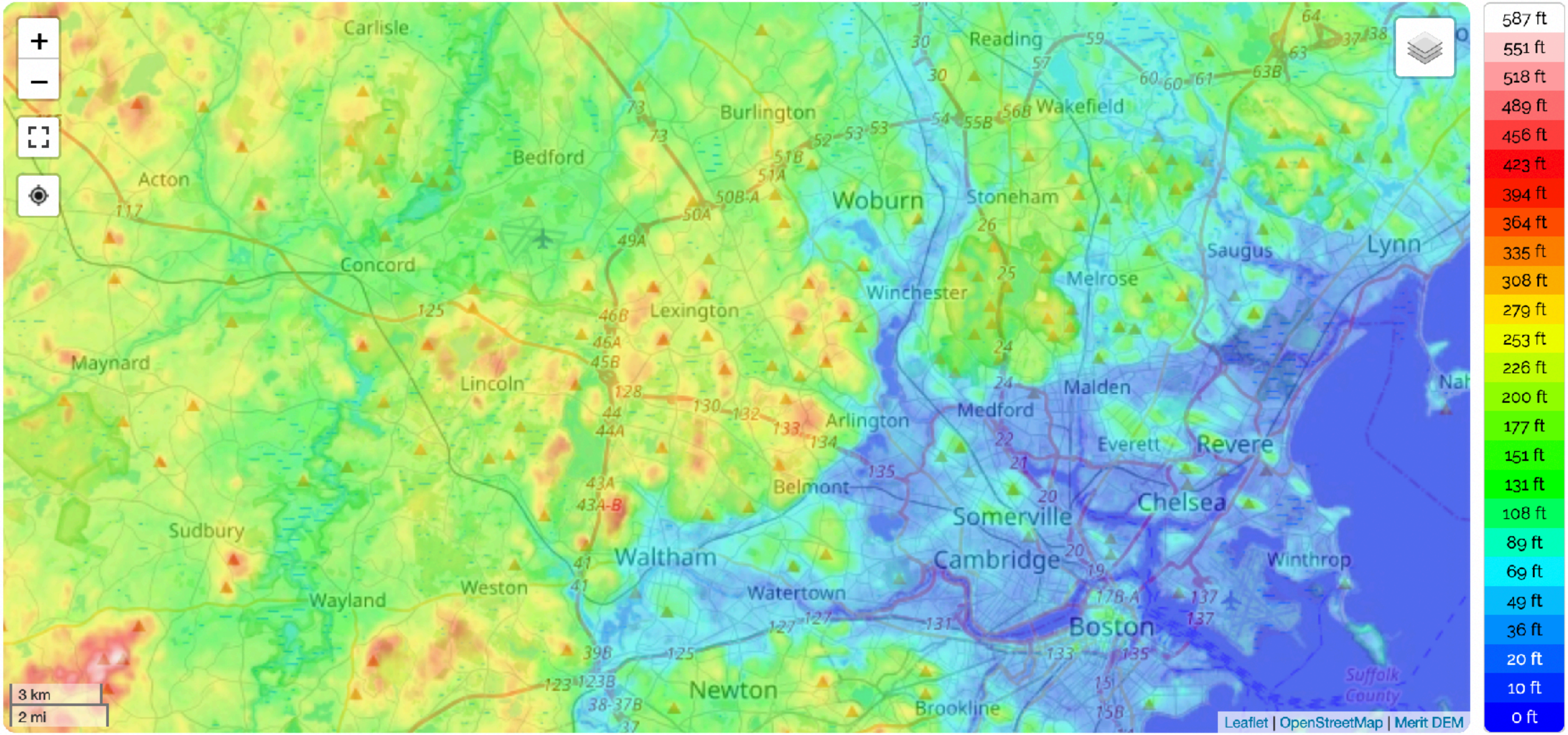


Pressure map



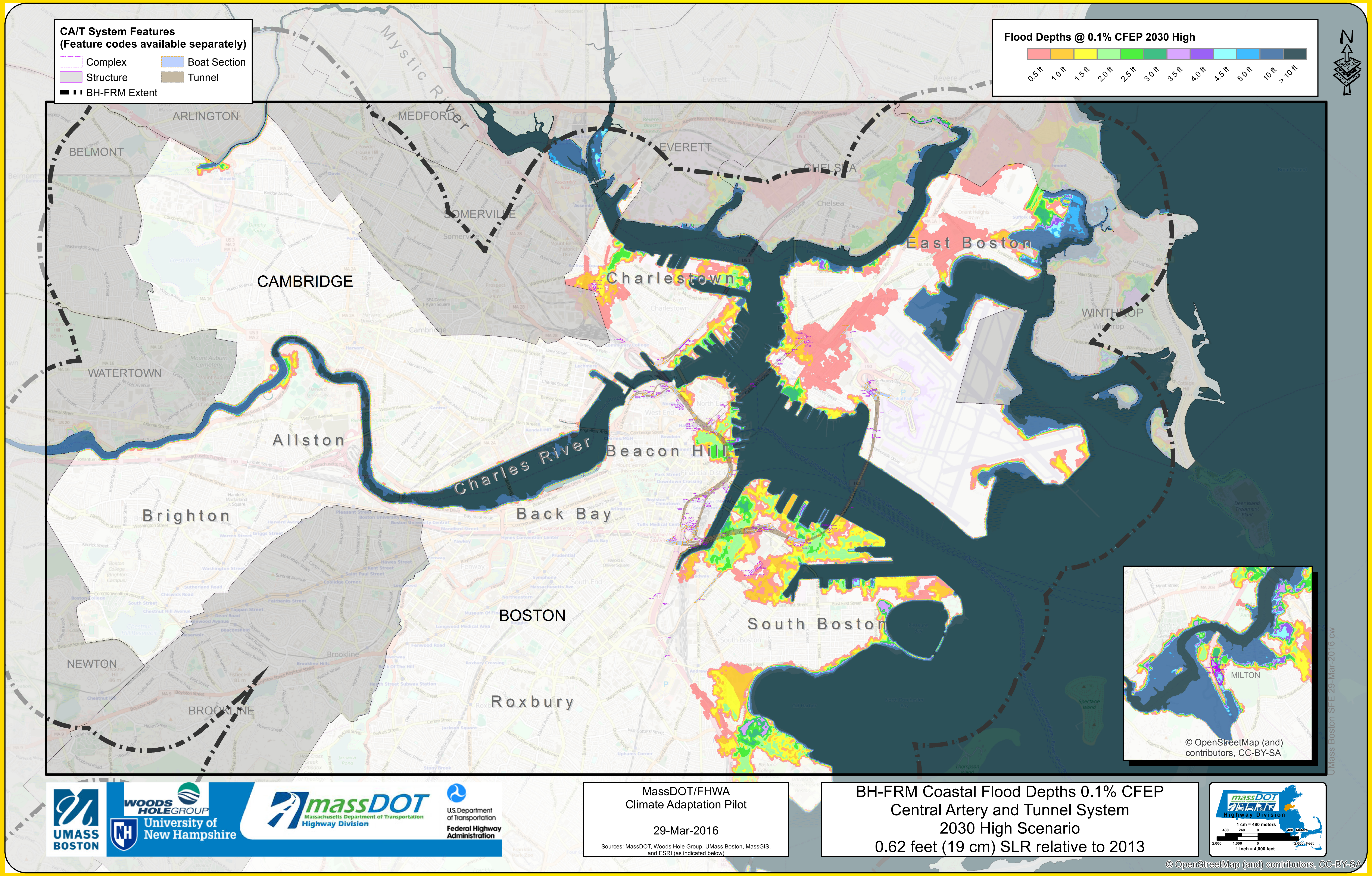
https://www.usairnet.com/weather/maps/current/barometric-pressure/

Elevation map



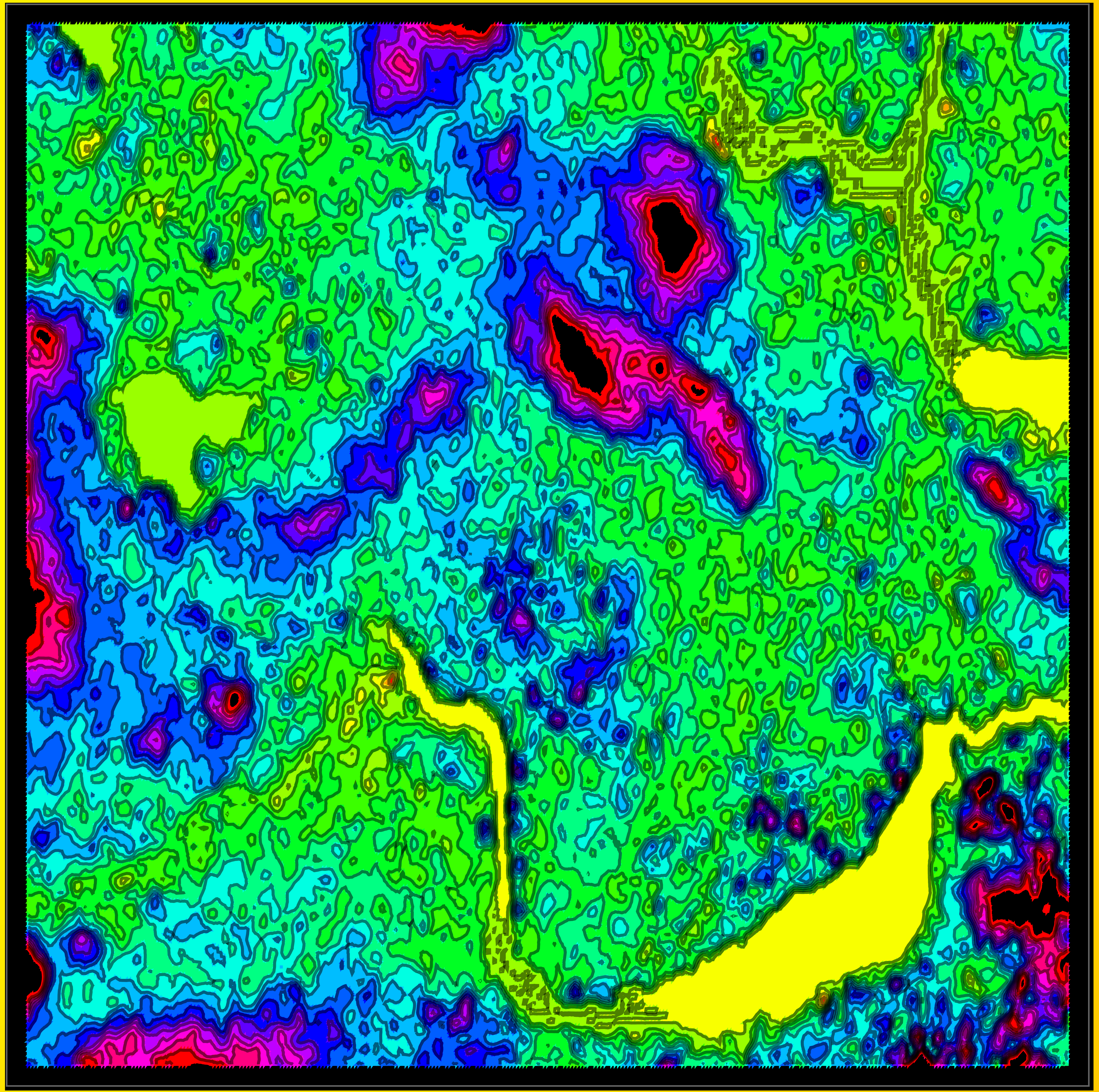
Boston, Suffolk County, Massachusetts, United States (42.36025 -71.05829)

Flood prediction



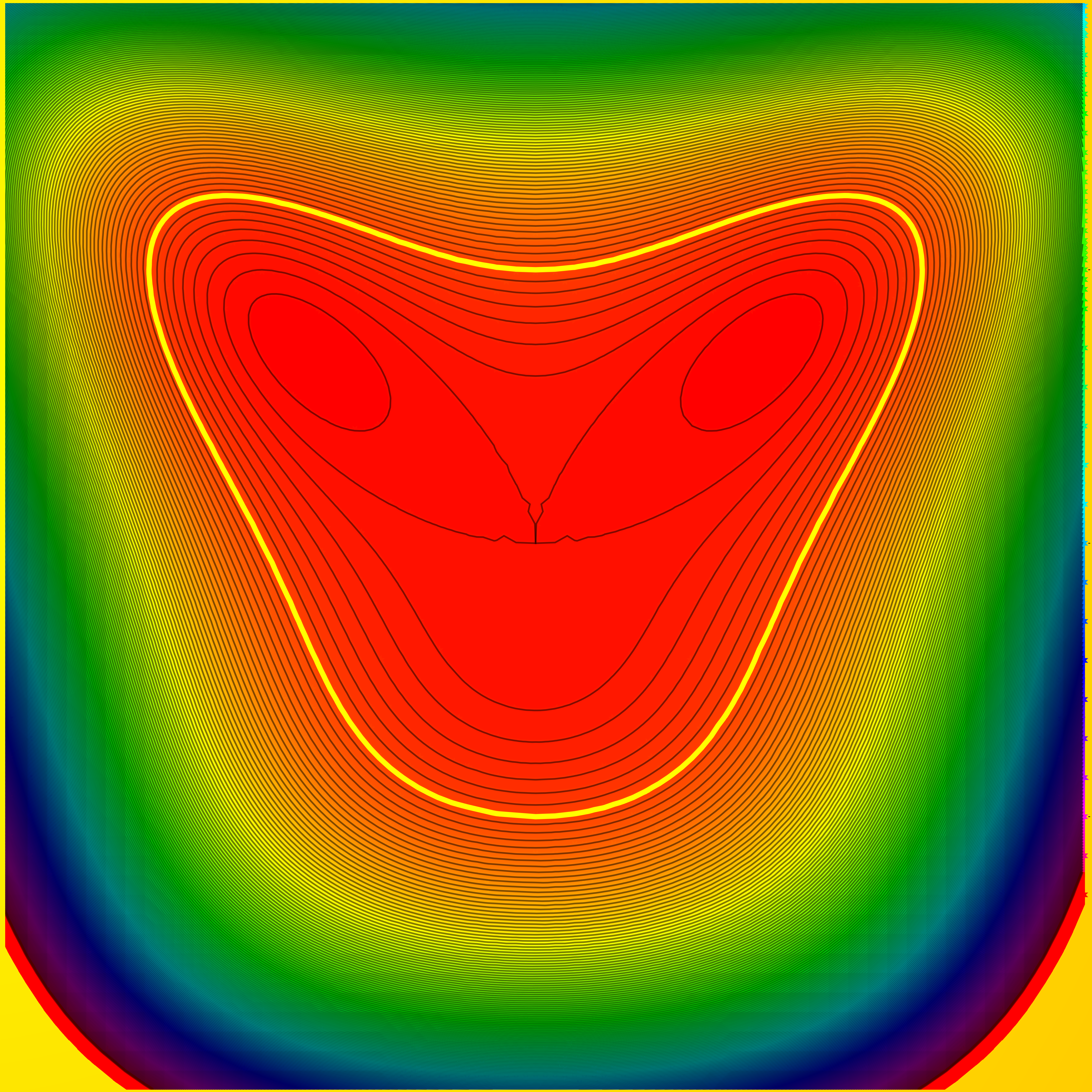
```
A = Reverse[ Normal[GeoElevationData[Entity["City",
 {"Cambridge", "Massachusetts", "UnitedStates"}], GeoProjection -> Automatic]]];
S = ListContourPlot[A, Contours -> 15, Background -> Black,
FrameTicks -> None, ColorFunction -> Hue]
```

Elevation map



Joker curve

$$x^4 + y^4 - 2x^2y = 1$$



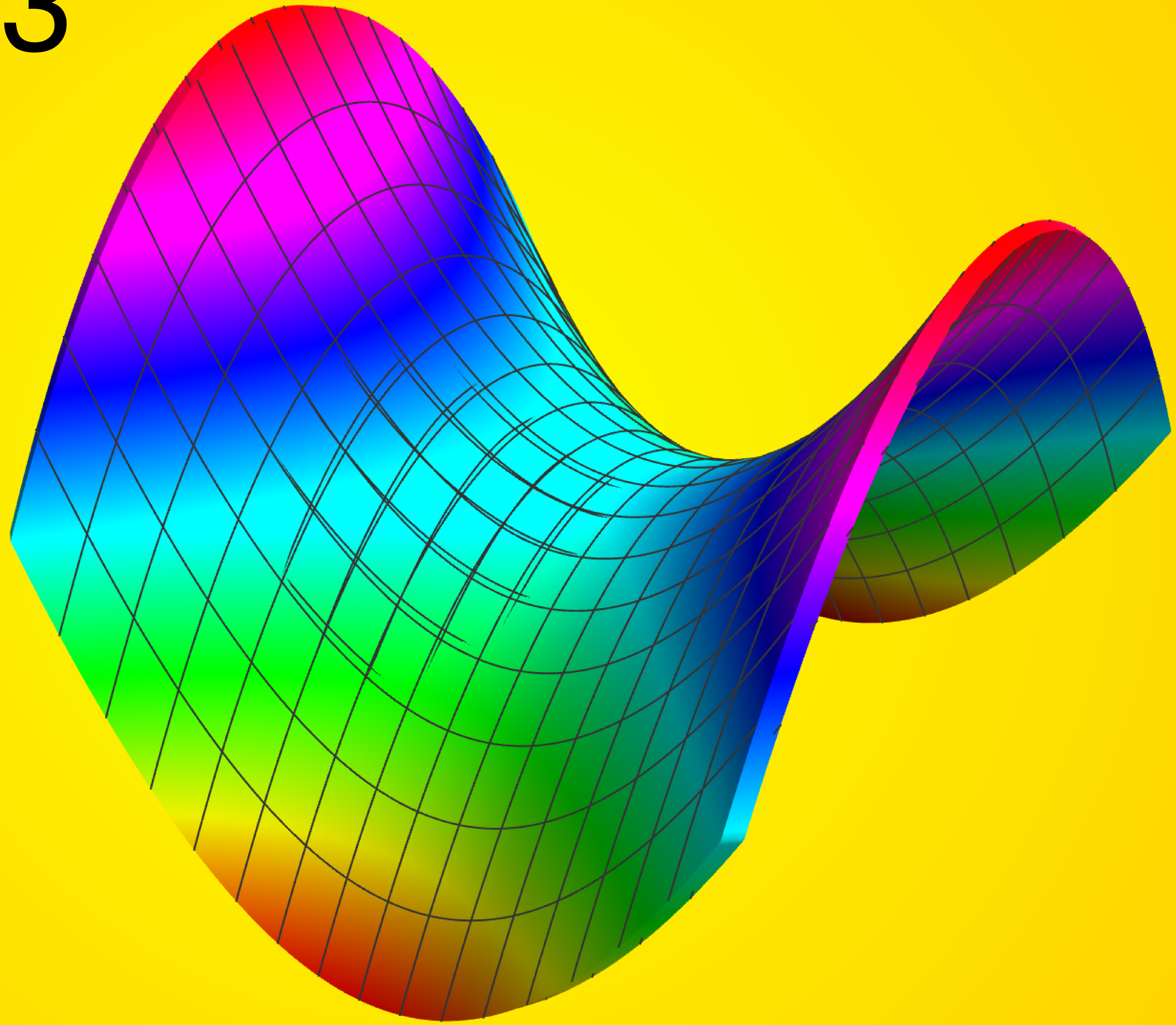
Top Gun 2022

The "crater"

Part 5

Worksheet part 3

Problem 3



Homework due Wednesday!

MQC will be closed Sunday

Happy Labor day

Problem Set 1 - Visualizing Functions of Two Variables

You should be able to:

- Sketch the contour map of a function of two variables, showing level sets with appropriate labels.
- Identify connections among a function, its graph, and its contour map.
- Explain the difference between the *graph* of a function and *level sets* of the function.

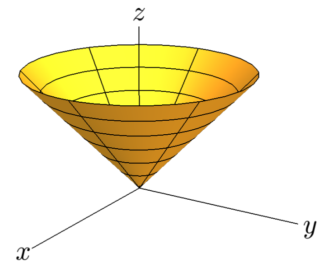
1. (a) Homework is an essential part of your learning in Math 21a. To learn how to get the most out of your homework assignments this semester, please read these tips.

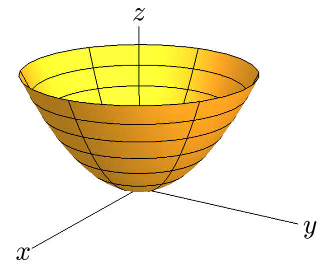
   (b) As you saw, one of the tips is to regularly discuss the material and problem sets with others. Here's a reminder of some ways to find others to talk math with. In the next week or so, please either talk with someone at the MQC or office hours, or get together with at least one other student outside of class. We'll ask you to write a little about your experience on Problem Set 6 (due Monday 9/19) and to share in class, so that you can hear from your peers about what different resources are like.

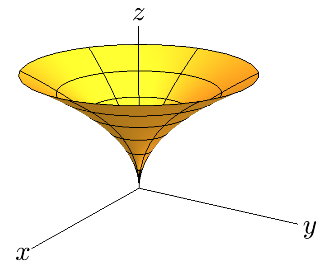
   (You don't need to turn in anything for this problem.)

2. (a) Draw a contour map of $f(x, y) = x^2 + y^2$ by drawing the level sets $f(x, y) = 0$, $f(x, y) = 4$, $f(x, y) = 8$, $f(x, y) = 12$, and $f(x, y) = 16$. Make sure each level set has the right x- and y-intercepts, and label each level set with the value of f.

   (b) Which of the following is the graph of $f(x, y) = x^2 + y^2$? Explain in a sentence.

   (I) 

   (II) 

   (III) 

   (c) Repeat (a) and (b) for $g(x, y) = \sqrt{4(x^2 + y^2)}$.

   (d) GeoGebra is a great online tool for visualizing in 3-D. Check your answer to (b) by going to `https://www.geogebra.org/3d` and typing

   ```
   z = x^2 + y^2
   ```

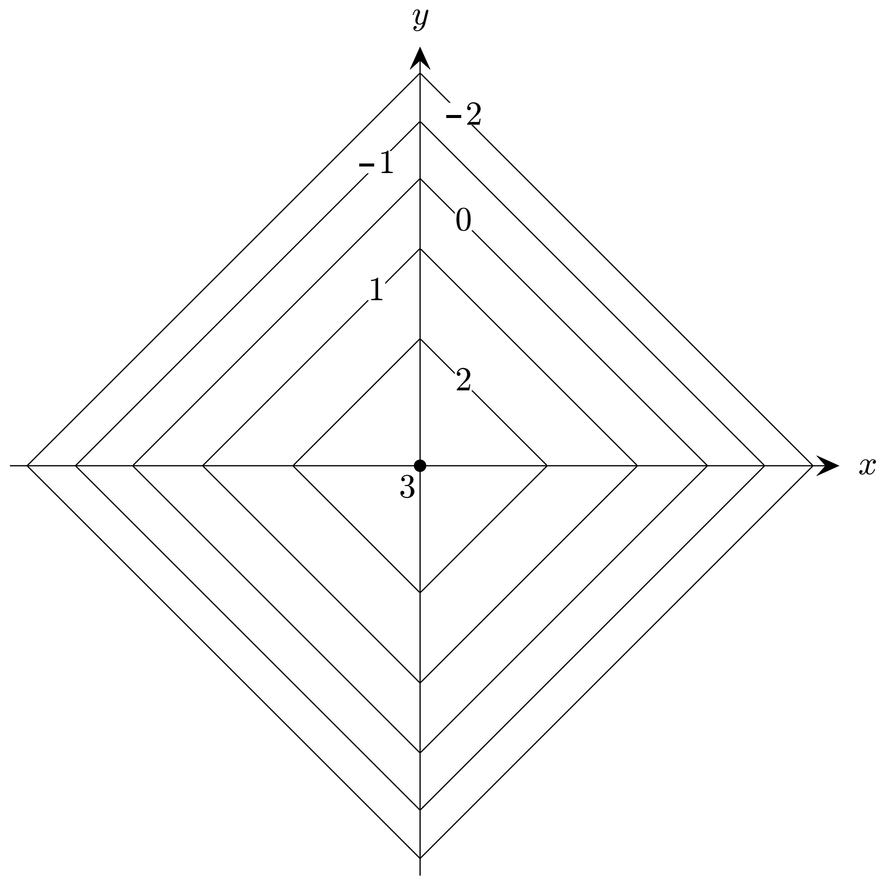
   One thing to watch out for is that the default orientation in GeoGebra is different from the one we use. To avoid getting tripped up by this, have GeoGebra label the axes: click the gear icon in the upper right corner, then select Settings. If you click on xAxis, there's an option to add a label to the x-axis, and similarly for the y- and z-axes. Label all 3 axes, and click and drag the picture to rotate it into our usual orientation. We encourage you to play around with the other settings.

   The shape of the graph $z = x^2 + y^2$ is called a paraboloid, and it's a shape we'll encounter regularly in Math 21a.

   Save a picture of your paraboloid and labeled axes by clicking the menu (≡) in the upper left corner and selecting Export Image, and submit it with your problem set.[(1)]

   If you'd like to get more familiar with GeoGebra, check out this video by former math preceptor Drew Zemke.

3. The contour map of a function $f(x, y)$ is shown, and the value of f on each contour is labeled. Use the contour map to make a rough sketch of the graph of f.



4. In this problem, real world phenomena are modeled by multivariable functions.

   (a) The pressure P inside a sealed balloon depends on the temperature T inside and the volume V. The pressure is proportional to temperature (higher temperature yields higher pressure), and inversely proportional to volume (higher volume yields lower pressure). Which of the following could be the graph of $P(T, V)$? Explain in a sentence.

(1) In case you need help putting this image into your PDF submission, here are some instructions:
- If you're using a Mac, you can use Preview to convert the image to a PDF and then combine PDFs.
- Otherwise, you can use this website to convert the image to a PDF and then this one to combine PDFs.

(c) A square-shaped farm lies in the region $0 \le x \le 5$, $0 \le y \le 5$, where x and y are measured in km. An irrigation pipe on the farm runs along the line $y = x$. The crop yield at position (x, y), measured in tons per km^2, is given by $f(x, y)$ and is higher closer to the pipe and lower further away. Which one of the following could be a formula for $f(x, y)$? Explain in a sentence.

i. $x - y$ ii. $|x - y|$ iii. $25 - (x - y)$ iv. $25 - (x - y)^2$

5. *In this problem, imagine that you're writing to a friend or relative who's never studied calculus. So, rather than using technical language like "differentiation", try to describe ideas in a way that's accessible to a general audience.*

(a) Give an example of a situation in which you'd want to model a quantity as a function of more than 1 variable; what variables might you want to use? Try to come up with an example that you personally would find interesting to study!

(b) When you studied single variable calculus, what are some questions you asked about functions of 1 variable?

(c) How might the questions you asked about functions of 1 variable be relevant to functions of more than one variable? Are there other questions you might ask about functions of more than one variable?

6. We're used to describing points in $\mathbb{R}^2$ by (x, y) coordinates and points in $\mathbb{R}^3$ by (x, y, z) coordinates, but these aren't the only way to describe points! Please read the "Polar Coordinates" handout to learn the basics of polar coordinates, a different coordinate system for $\mathbb{R}^2$ that we'll build on next time. Then do the following problem.

(a) Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the same point.

i. $(r, \theta) = (3, \frac{\pi}{4})$ ii. $(r, \theta) = (4, \frac{2\pi}{3})$

(b) In each part, sketch the points in $\mathbb{R}^2$ whose polar coordinates satisfy the given equations / inequalities. Please mark any important values on your axes.

i. $r = 3$ and $-\pi \le \theta \le 0$

ii. $1 \le r \le 4$ and $\frac{\pi}{2} \le \theta \le \frac{7\pi}{6}$

For next class, skim the following excerpts from OpenStax *Calculus Volume 3*: the part of §2.2 between Checkpoint 2.13 and Checkpoint 2.17, and §2.7 up to Checkpoint 2.57.

7. (Optional extra credit)

A machine learning application of distance between points. Machine learning is a branch of artificial intelligence in which computers "learn" to perform some task; a common task is to *classify* data into categories. In this problem, we'll look at one example.

This data set contains data about 400 patients, including *attributes* such as their age (in years), blood glucose (in milligrams per deciliter), and hemoglobin (in grams per deciliter). The data set also includes whether each patient has chronic kidney disease (CKD). We call this the *training data set* because we'd like to use this data to train a computer to predict whether new patients (who aren't in this data set) have CKD.

(a) One simple method for doing this is called *nearest neighbor*. It works like this: for each patient in the training data set, visualize the ordered triple (age, blood glucose, hemoglobin) as a point in $\mathbb{R}^3$. This applet shows this visualization; each point is colored red if the patient has CKD and green if they don't.

If the computer is given a new point representing information about a new patient, it simply finds the point in the training data set that's closest to the new point, and then gives the new patient the same diagnosis as that closest point (so, if the closest point is red, the computer will predict the new patient has CKD; if the closest point is green, the computer will predict the new patient doesn't have CKD).

A hospital would like your assessment of how well this method will work for diagnosing CKD. Looking at the visualization of the training data set, what would you tell them? Be sure to explain your reasoning.

(b) Suppose that each patient's hemoglobin had been expressed in milligrams per deciliter rather than in grams per deciliter. How would that affect the results of the nearest neighbor method?

(c) In this example, we can visualize the training data as points in $\mathbb{R}^3$. So, to carry out the nearest neighbor method, we just need to be able to find the distance between two points in $\mathbb{R}^3$.

The idea of $\mathbb{R}^2$ and $\mathbb{R}^3$ can be generalized; for any positive integer n, we define $\mathbb{R}^n$ to be the set of n-tuples $(a_1, \ldots, a_n)$.

Explain why it could be useful in machine learning to find the distance between two points in $\mathbb{R}^n$ for $n > 3$.

(d) We can use geometry to find the distance between two points in $\mathbb{R}^2$ or in $\mathbb{R}^3$. We can't visualize $\mathbb{R}^n$ for $n > 3$, so we instead define the distance between two points in $\mathbb{R}^n$ to be consistent with what happens when $n = 2$ or $n = 3$. How do you think we should define the distance between two points $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ in $\mathbb{R}^n$? (Give a formula, and explain.)

THE END