Lecture 2

Dimension
Polar coordinates
Cylindrical coordinates

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Start with Worksheet

Dimension

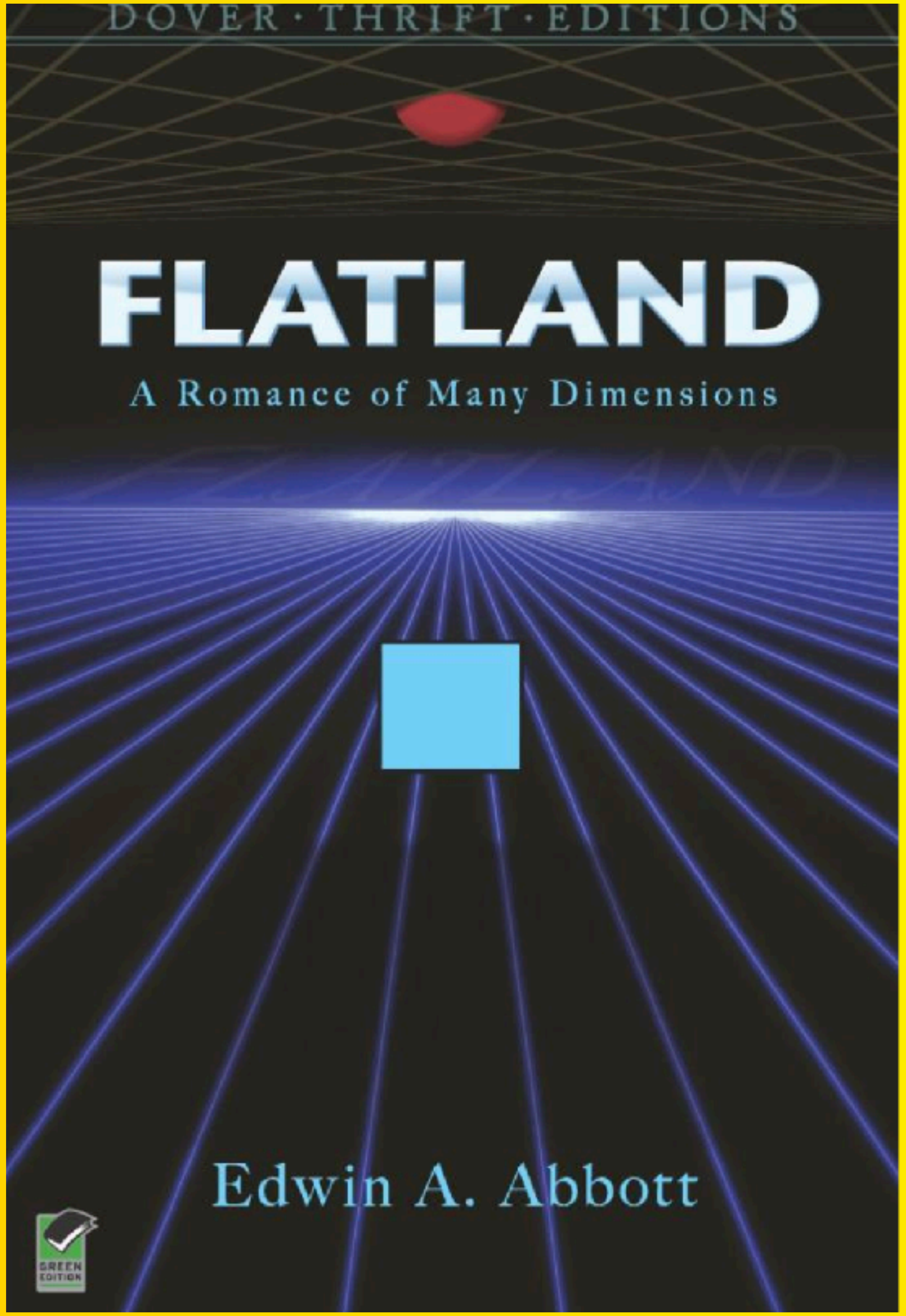
What is the dimension?

		$\mathbb{R}^2$	$\mathbb{R}^3$
A	$y = x^2$		
B	$x^2 + y^2 = 0$		
C	$x^2 + y^2 = 1$		
D	$x + y = 1, x - y = 3$		

About ambient space

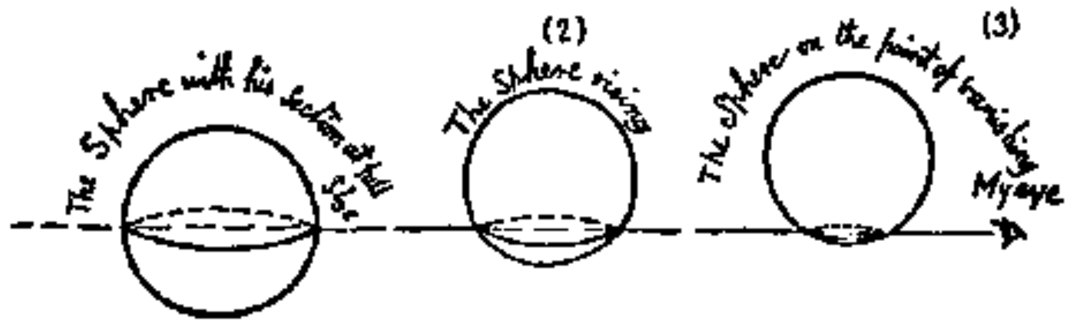
Why is our space three dimensional?

Two dimensional space.



because that Linear Realm had not Dimensions enough to represent the whole of you, but only a slice or section of you? In precisely the same way, your country of Two Dimensions is not spacious enough to represent me, a being of Three, but can only exhibit a slice or section of me, which is what you call a Circle.

The diminished brightness of your eye indicates incredulity. But now prepare to receive proof positive of the truth of my assertions. You cannot indeed see more than one of my sections, or Circles, at a time; for you have no power to raise your eye out of the plane of Flatland; but you can at least see that, as I rise in Space, so my sections become smaller. See now, I will rise; and the effect upon your eye will be that my Circle will become smaller and smaller till it dwindles to a point and finally vanishes.



There was no "rising" that I could see; but he diminished and finally vanished. I winked once or twice to make sure that I was not dreaming. But it was no dream. For from the depths of nowhere came forth a hollow voice - close to my heart it seemed - "Am I quite gone? Are you convinced now? Well, now I will gradually return to Flatland and you shall see my section become larger and larger."

Every reader in Spaceland will easily understand that my mysterious Guest was speaking the language of truth and even of simplicity. But to me, proficient though I was in Flatland Mathematics, it was by no means a simple matter. The rough diagram given above will make it clear to any Spaceland child that the Sphere, ascending in the three positions indicated there, must needs have manifested himself to me, or to any Flatlander, as a Circle, at first of full size, then small, and at last very small indeed, approaching to a Point. But to me, although I saw the facts before me, the causes were as dark as ever. All that I could comprehend was, that the Circle had made himself smaller and vanished, and that he had now reappeared and was rapidly making himself larger.

When he regained his original size, he heaved a deep sigh; for he perceived by my silence that I had altogether failed to comprehend him. And indeed I was now inclining to the belief that he must be no Circle at all, but some extremely clever juggler; or else that the old wives' tales were true, and that after all there were such people as Enchanters and Magicians.

After a long pause he muttered to himself, "One resource alone remains, if I am not to resort to action. I must try the method of Analogy." Then followed a still longer silence, after which he continued our dialogue.

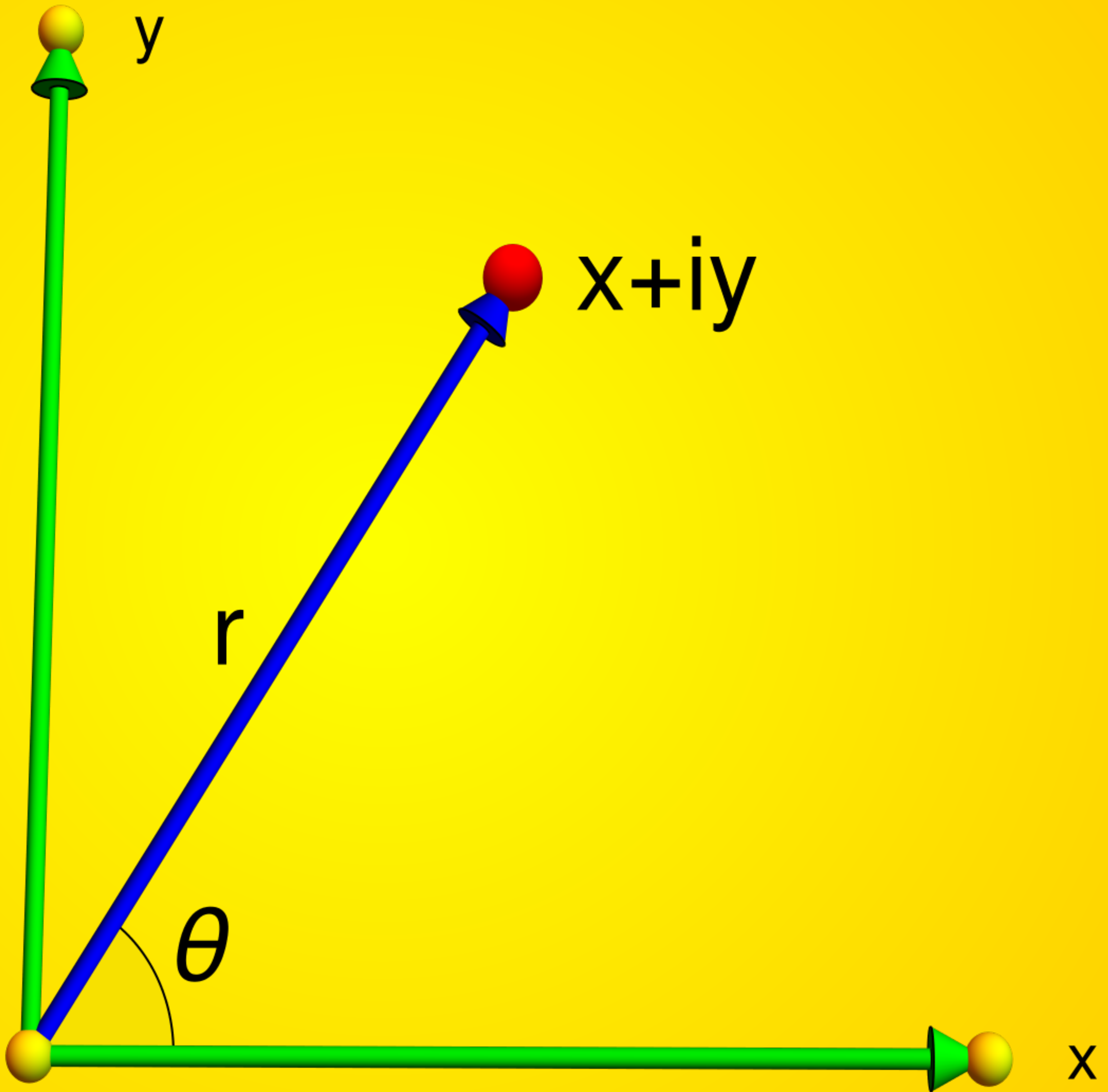
Sphere. Tell me, Mr. Mathematician; if a Point moves Northward, and leaves a luminous wake, what name would you give to the wake?

I. A straight Line.

Spheres!

Worksheet problem 4

Polar Coordinates


y
x+iy
r
θ
x

Some Objects

A

$$r = \theta$$

B

$$\theta = 1$$

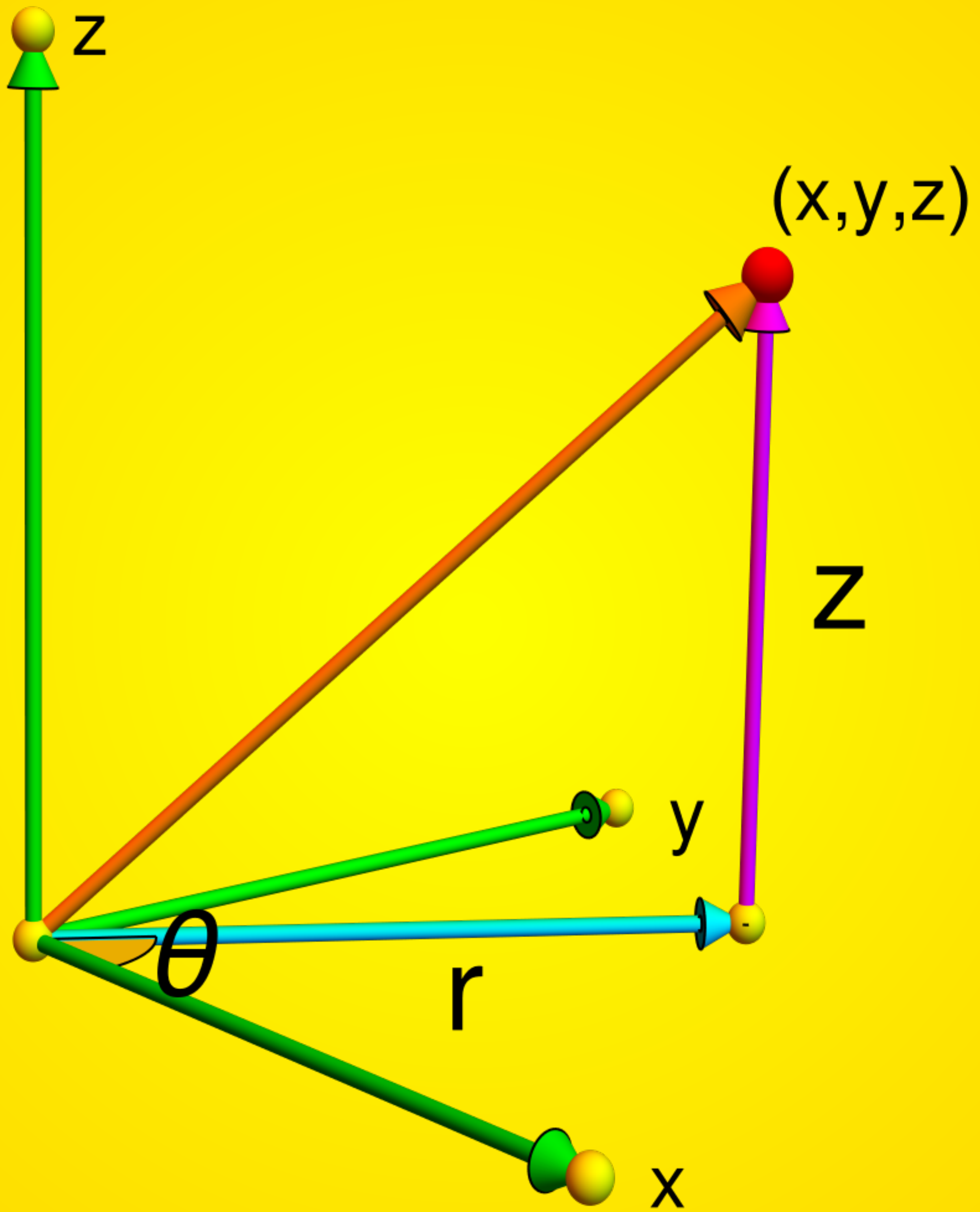
C

$$r = \cos(\theta)$$

D

$$r = 1$$

Cylindrical Coordinates


z
(x,y,z)
z
y
θ
r
x

Some Objects

A $r = \theta$

B $\theta = 1$

C $r = \cos(\theta)$

D $r = z$

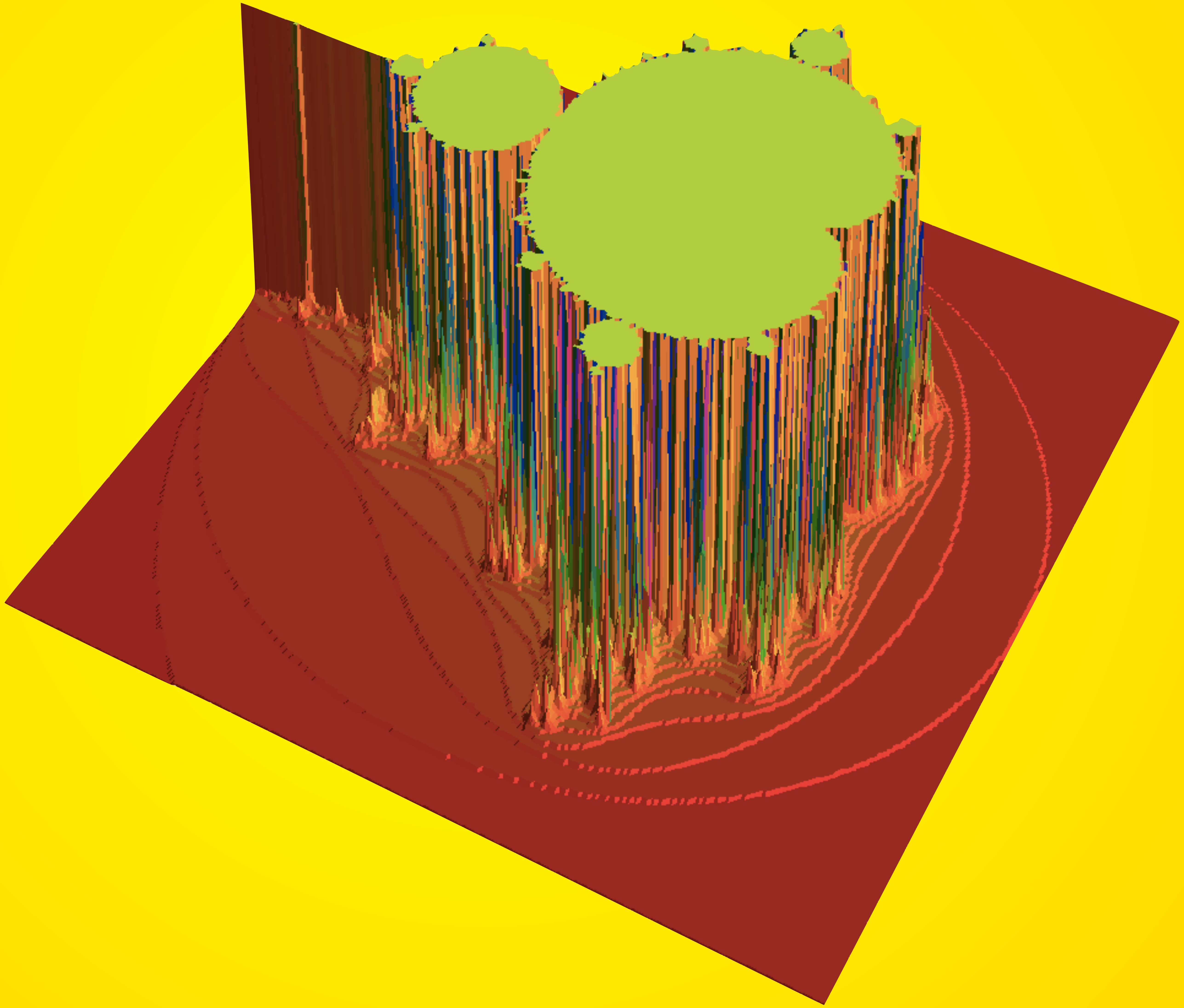
Worksheet problems

5–6

Illustrations

Related to Polar


map of the
MANDELBROT SET
Defined as the set of points in the complex plane that do not escape to infinity under iterations of
$z_{n+1} = z_n^2 + c$
where z and c are complex numbers, $z_0 = 0$, and c is determined by the initial position in the complex plane. Rendered with 9,699,690 iterations, using distance estimation to draw the boundary.
mapped by Bill Tavis
additional illustrations Lena Martin
set contains many patterns, some of which become quite complex when the boundary is magnified. The path taken when zooming determines the characteristics of the pattern. For example, the local bulb numbering can be observed by counting the number of branches at the tip of (and including) the local needle. The densely covered region to the right was found by zooming to the tip of local bulb 6 of local bulb 5 of local bulb 4 of bulb 3 on the main cardiod. The resulting period of the bulb is then 360, the product of 6·5·4·3·1.
720
360
(below) The largest bulb between any two bulbs has a period equal to the sum of the first two. In this way, the Fibonacci sequence can be found. Between 2 and 3, the largest bulb is period 5; between 3 and 5, the largest bulb is period 8; and after that, the largest bulbs are 13, 21, 34, 55, etc.
Period 3 Bulb
Triple Spiral Valley
bulb 3 attaches c=-0.125+0.6495i
Spirals (above) and nets (left) are patterns that can be found within every valley.
Zooming in on an arm of the spiral, it is composed of a complex, yet specific, pattern. Clustered together are more spirals along with nets. Some of the spirals are nearly identical to the larger one and others double near the center.
The Needle begins at c=-1.401155, where the bulbs in the west of the Main Cardiod continue to double their periods until reaching infinity. The view below (only 10^{-11} across) shows how the branches continue to pile up. In the limit, they will become infinitely dense at this point, known as the Myrberg-Feigenbaum point.
512
256
Lightning Branch
Lightning minibrot
The branches coming off of the bulbs increase in number as they go into the valleys, forming spirals and nets (upper right).
Orbits in the west side of the Main Cardiod have star-shaped patterns. This example orbit, for c=-0.45+0.47i, creates a five-pointed star because it begins beneath the Period 5 bulb.
bulb 4 attaches c=0.25+0.5i
Orbits in the east side of the Main Cardiod have spiral-shaped patterns. The example orbit above, for the point c=0.15+0.4i, creates a four-sided spiral because it originates beneath the Period 4 bulb.
The example orbit below, for the point c=0.285+0.05i, creates a smooth logarithmic spiral inward, exiting and reentering the set.
Axial Cusp
c=0.25
Imaginary
Real
Period 2
Main
Bulb
Sceptor Valley
Double Spiral Valley
The Mandelbrot set is a geometric shape much like a circle or triangle. It differs in that the boundary is infinitely complex, revealing endless detail upon magnification. (see inset)
The Needle
The Mandelbrot set is perfectly symmetrical about the Real axis.
Period 1
Main
Inside of Set
Cardioid
Seahorse Valley
Elephant Valley
Complex numbers have a real part and an imaginary part, represented as a + bi, where a is the real part, b is the imaginary part, and i is the square root of -1. A complex number can then be located in the plane by forming the coordinate point (a,b). To square a complex number, it is written (a+bi)·(a+bi), which becomes a^2+abi+bai-b^2, simplifying to the new complex number of (a^2-b^2)+(2ab)i. Testing a point to see if it belongs in the Mandelbrot set is done by squaring the current value and adding a constant value, which is determined by the initial location in the complex plane. After many iterations, if the point is at a distance of less than 2 from the origin, it is considered part of the set.
Julia Medallions
Medallions can be found along the extra tendrils that grow off every minibrot, and are adorned with patterns inherited from the minibrot. For example, the medallions below have lightning-shaped tendrils inherited from the Lightning minibrot around which they are found (locations indicated by color). The medallions resemble the Julia sets that correspond to their location. Julia sets are fractals that use the same equation as the Mandelbrot set, but differ in that c is constant for the entire image and z_0 varies with the initial position in the complex plane.
Elephant Medallion
Brain Medallion
Branch Medallion
Triple Spiral Medallion
Lightning Minibrot
Seahorse Medallion
Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.
– Benoit Mandelbrot, father of fractal geometry
Minibrot Islands
The border of the Mandelbrot set contains an infinite number of minibrots - small islands that resemble the whole, surrounded by intricate patterns unique to each island. This self-similarity across many scale levels is an important characteristic of fractal geometry. The minibrot to the left was found at the center of an elephant medallion from a tendril of the
117
258
Period 3 Bulb
Triple Spiral Valley



Placed in google earth.


10^{-0}

Homework due Friday

Problem Set 2 - Dimension, Introduction to Cylindrical Coordinates

You should be able to:

- Identify the dimension of an object. State and use the rule of thumb relating the dimension of an object to the number of equations defining it.
- Explain the difference between the *graph* and *level sets* of a function.
- Write and interpret equations for spheres in $\mathbb{R}^3$.
- Describe the geometric meaning of r and θ in polar and cylindrical coordinates.

In Math 21a, you're welcome to use GeoGebra on your homework unless we specifically ask you to do a problem without using technology. Even then, you're welcome to check your answer using GeoGebra; you just need to explain how you can solve the problem without it. We encourage you to get familiar with GeoGebra; it's a great tool that will help you throughout the semester.

1. Write up your answers to Workshop 1, #1 and #2. (In #1(a) and #1(d), you can describe in words where the points are rather than marking them.)

2. In class, we talked about the idea of dimension; please read the "Dimension" handout, which fleshes this idea out.

   (a) Imagine that you're tutoring a student, and the student says to you,

   "I don't understand why a sphere is considered to be 2-dimensional. After all, a sphere isn't flat. Shouldn't it be 3-dimensional?"

   How would you explain to the student why a sphere is 2-dimensional?[1]

   (b) In each part, sketch the object described, and say what its dimension is.

   i. The part of $x^2 + y^2 + z^2 = 3$ below $z = 1$.

   ii. The part of $z = 1$ inside $x^2 + y^2 + z^2 = 3$.

   iii. The intersection of $x^2 + y^2 + z^2 = 3$ and $z = 1$.

   (c) One of the three objects in (b) is 1-dimensional; in other words, it's a curve. What's the length of this curve? (We don't know how to calculate length in general, but this curve is a very familiar shape.)

   (d) Two of the three objects in (b) are 2-dimensional; in other words, they're surfaces. Although we don't know how to find surface area in general yet, you should be able to find the surface area of one of these two surfaces (because it's a simple geometric shape). Do this.

3. (a) Sketch the shape in $\mathbb{R}^2$ defined by $x^2 + y^2 = 9$.

[1] Research has found that people learn material better when they teach it (as opposed to simply studying it). In fact, just preparing to teach the material has a beneficial effect on learning (even if you never actually teach it to someone). So, even when we don't explicitly ask you to imagine teaching the material to someone, it's worthwhile to give it a try!

   (b) In $\mathbb{R}^3$, what shape do you think the equation $x^2 + y^2 = 9$ describes? Write down what you think; you'll get full credit even if it isn't right.

   (c) Does your guess in (b) agree with the rules of thumb in the "Dimension" handout? Why or why not?

   (d) Use GeoGebra to see what $x^2 + y^2 = 9$ looks like, and sketch the result.

   (e) Imagine you're tutoring a student, and the student says to you,

   "I don't understand why the shape of $x^2 + y^2 = 9$ in $\mathbb{R}^3$ looks like this."

   How would you explain this to the student?

   (f) What shape does $y^2 + z^2 = 4$ describe in $\mathbb{R}^3$? Sketch it without using technology, but then feel free to check your answer using GeoGebra. (Remember that the way GeoGebra orients its axes can be confusing; see Problem Set 1, #2(d) for a reminder of how to deal with this.)

4. In class, you talked about visualizing functions of 3 variables. If you want a refresher, watch this quick video.

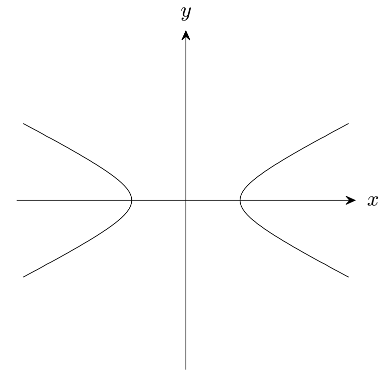
   (a) Which picture below shows a level set of $f(x, y, z) = 4x^2 + y^2$? (Notice that f is a function of 3 variables!) Explain briefly.

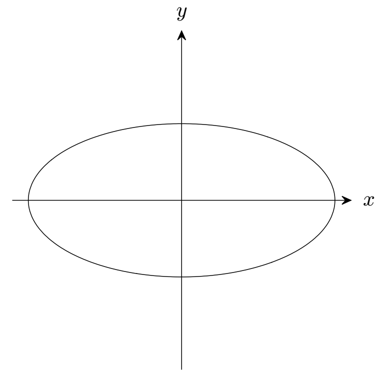
   (b) Which picture shows the graph of $g(x, y) = 4x^2 + y^2$? (Notice that g is a function of 2 variables!) Explain briefly.

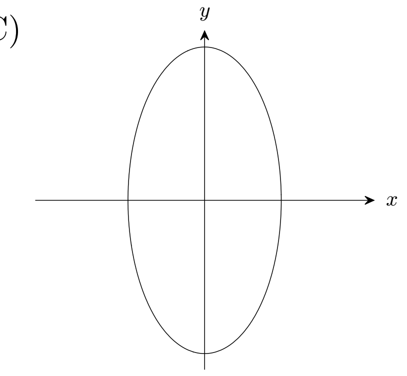
   (c) Which picture shows a level set of $g(x, y) = 4x^2 + y^2$? Explain briefly.

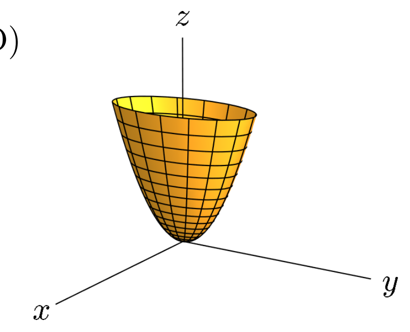
   (d) Explain in complete sentences why (a) and (c) are different.

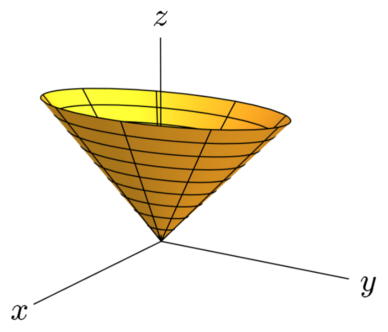
Choices:

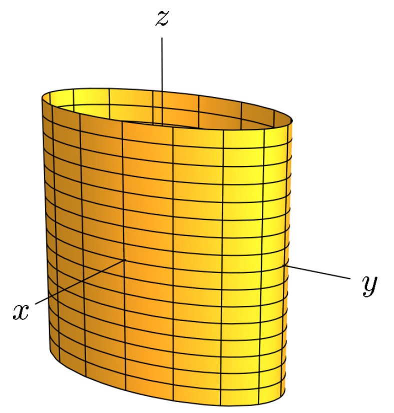
(A)


(B)


(C)


(D)


(E)


(F)


5. Go to `https://www.geogebra.org/m/ejwq7f7f`; you should see a solid. The two curved sides of the solid are parts of cylinders centered around the z-axis.

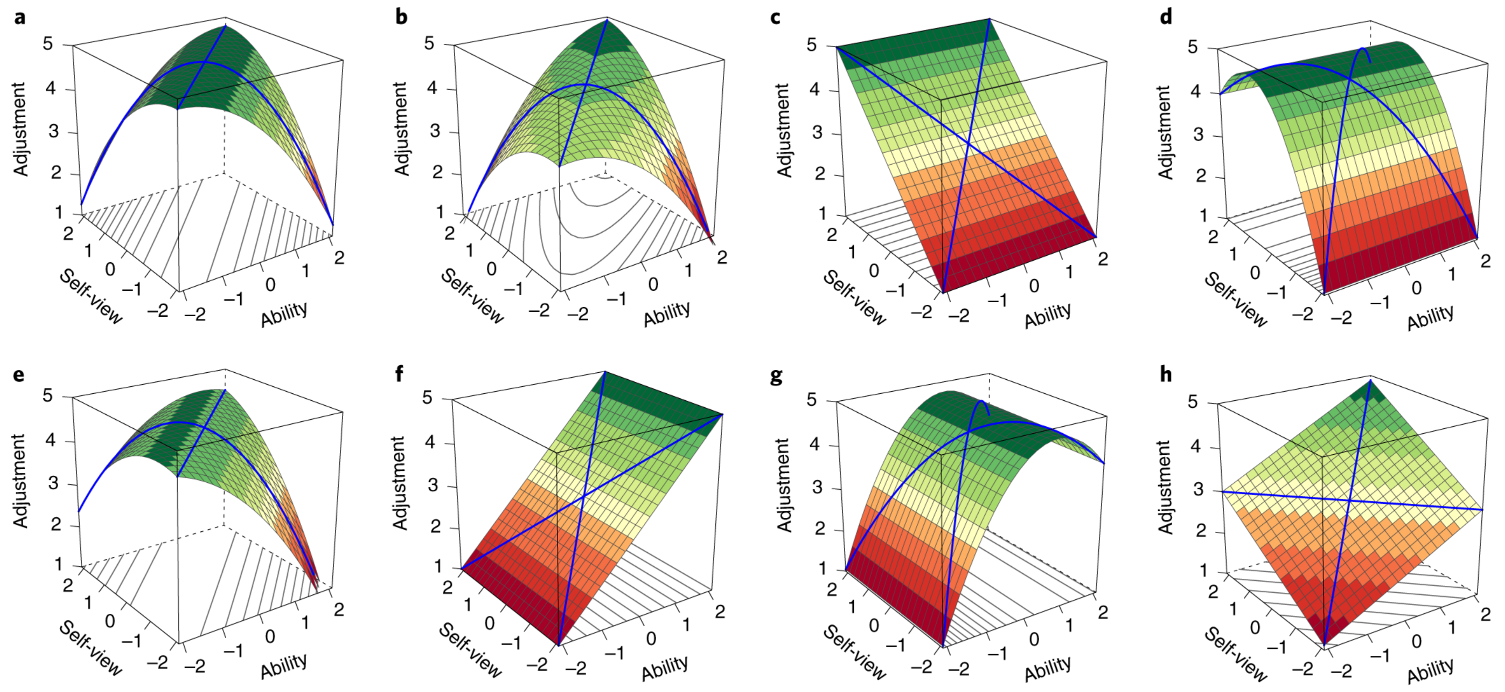
   Fill in the blanks in the statement below; no explanation is necessary.

   The solid shown can be described in cylindrical coordinates by

   $$\_\_\_ \le r \le \_\_\_, \quad \_\_\_ \le \theta \le \_\_\_, \quad \_\_\_ \le z \le \_\_\_.$$

6. *In 21a, we're used to drawing our coordinate axes in a particular orientation, but not all sources use the same orientation. This problem is an example of that.*

   The psychology study [He and Côté, 2019] tested different hypotheses on how people's psychological "adjustment" (how satisfied they are with their life, career, and relationships) depends on their abilities and "self-view" (how they assess their own abilities). Here's Figure 1 from the paper:



   This figure shows graphical representations of 8 hypotheses the authors considered; look carefully at how the axes are numbered.

   (a) What are the black lines / curves at the bottom of each box? What do the colors on each graph represent?[2]

   (b) One hypothesis the authors considered was that adjustment depends only on self-view, not on actual ability, and higher self-view corresponds to higher adjustment. Which one of the graphs above (a - h) represents this hypothesis?

   (c) Another hypothesis the authors considered was that adjustment depends only on how closely people's abilities and self-view match, and that "individuals are optimally adjusted when their self-views and abilities match." (In this hypothesis, people's actual abilities make no difference.) Which one of the graphs above represents this hypothesis?

For next class, read OpenStax *Calculus Volume 3* - §2.7.

[2] If you have trouble seeing the colors in this image, try this version.

THE END