Lecture 3

Spherical Coordinates

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Dimension Review

dim=1

Circle

S_1

2π

Sphere

S_2

4π

dim=2

Disc

B_2

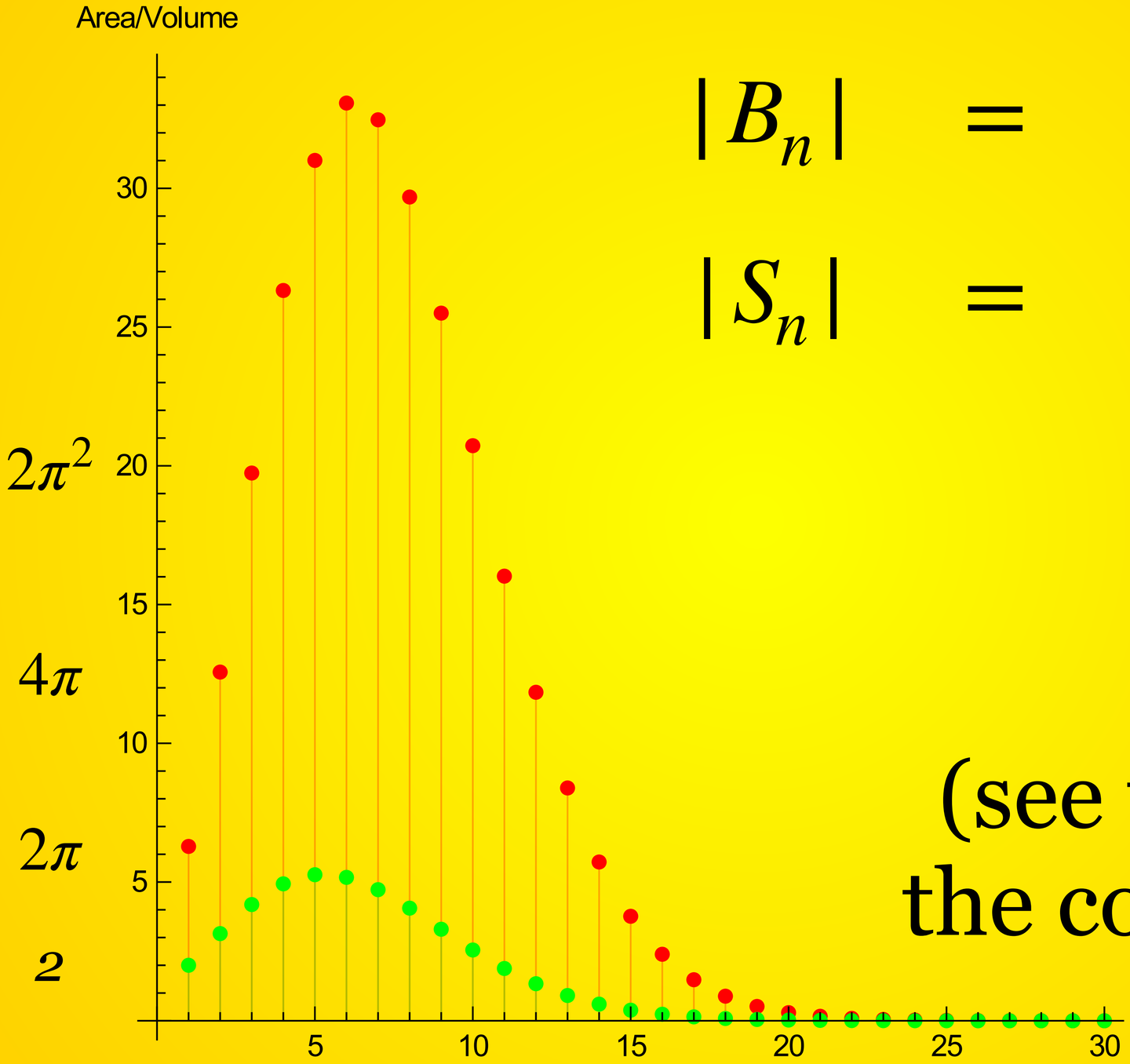
π

dim=3

Ball

B_3

$4\pi/3$



$$|B_n| = \frac{2\pi}{n}|B_{n-2}|$$

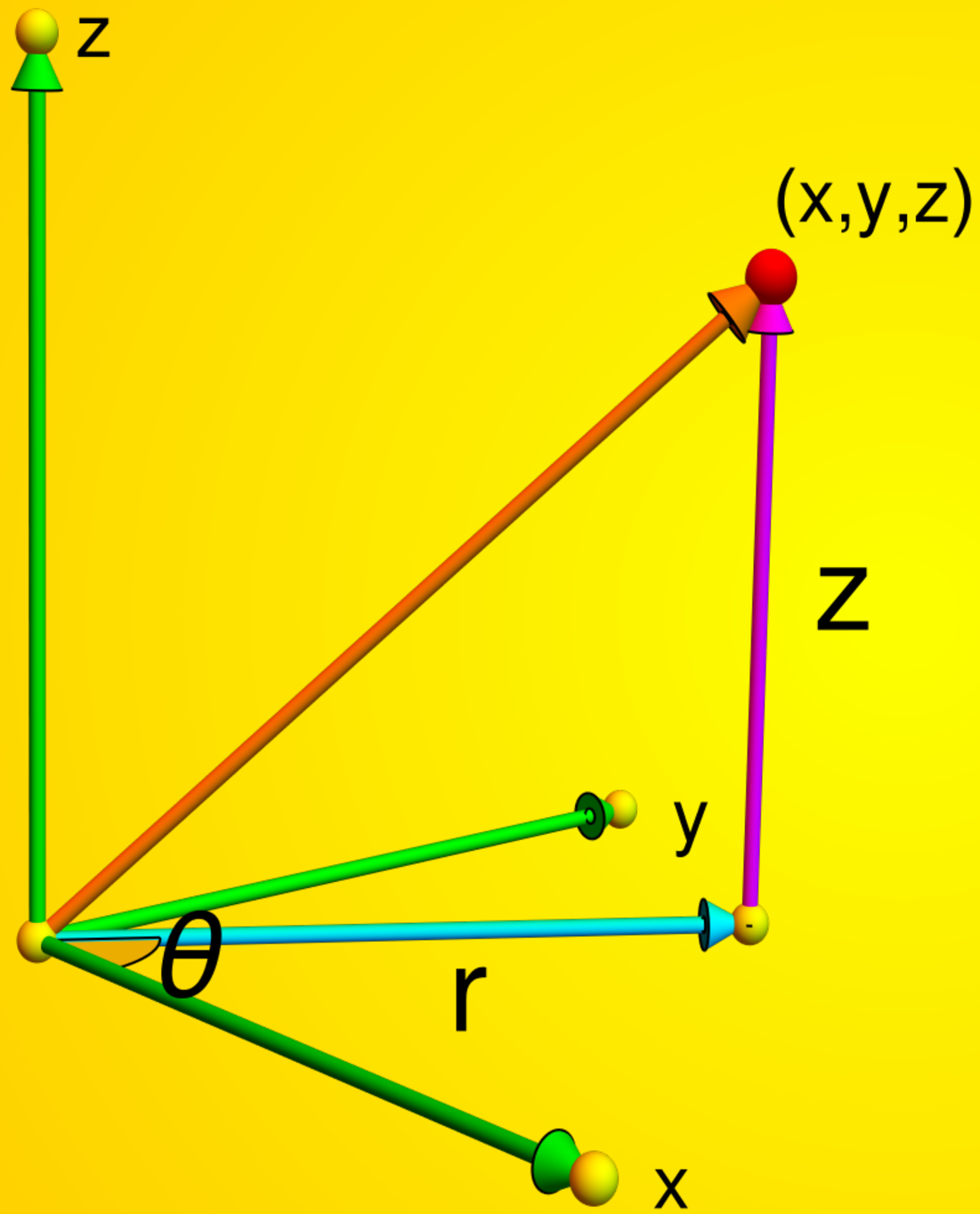
$$|S_n| = \frac{2\pi}{n-1}|S_{n-2}|$$

Spheres

Balls

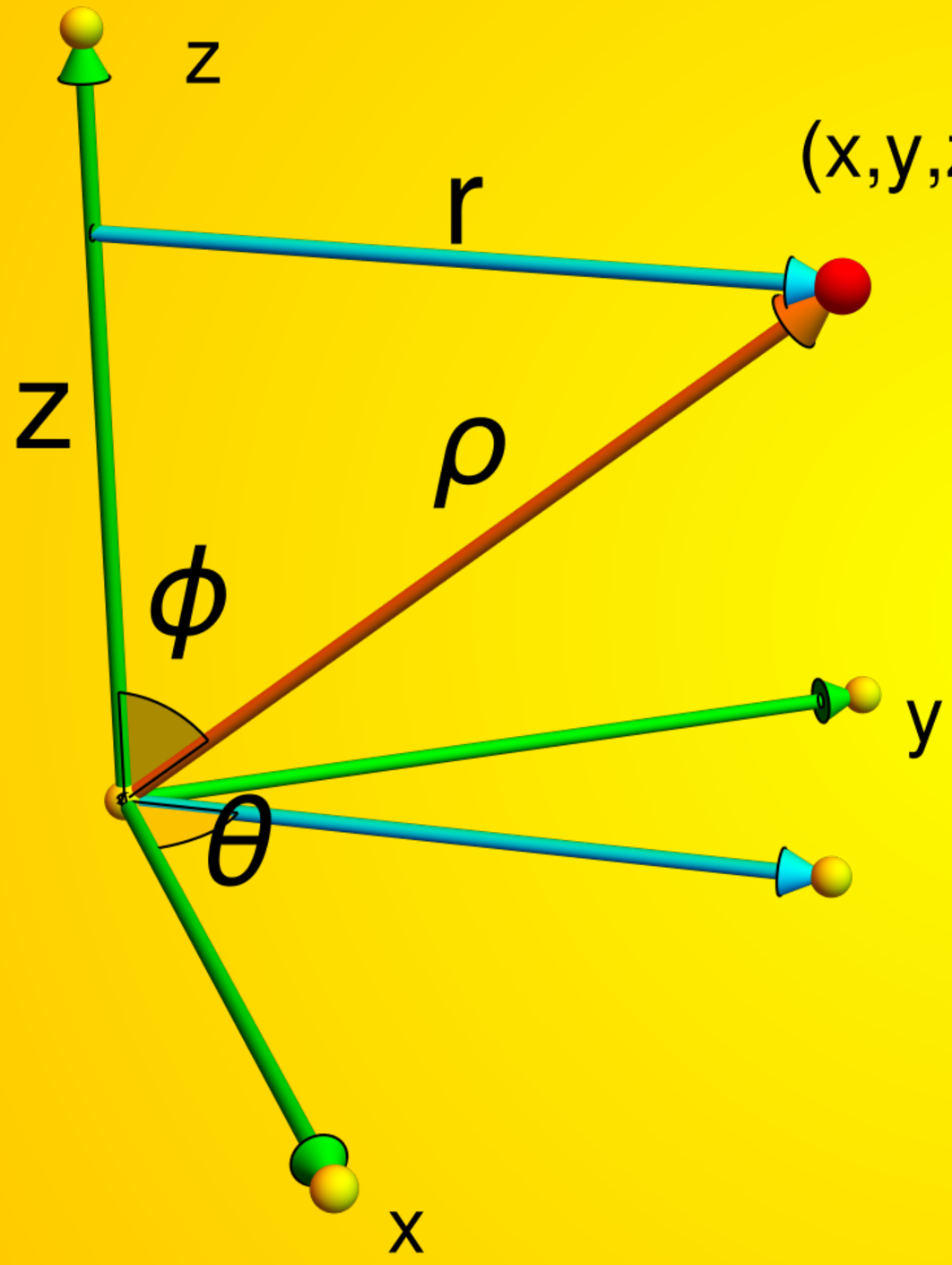
(see the movie on the course website)

Cylindrical Coordinates



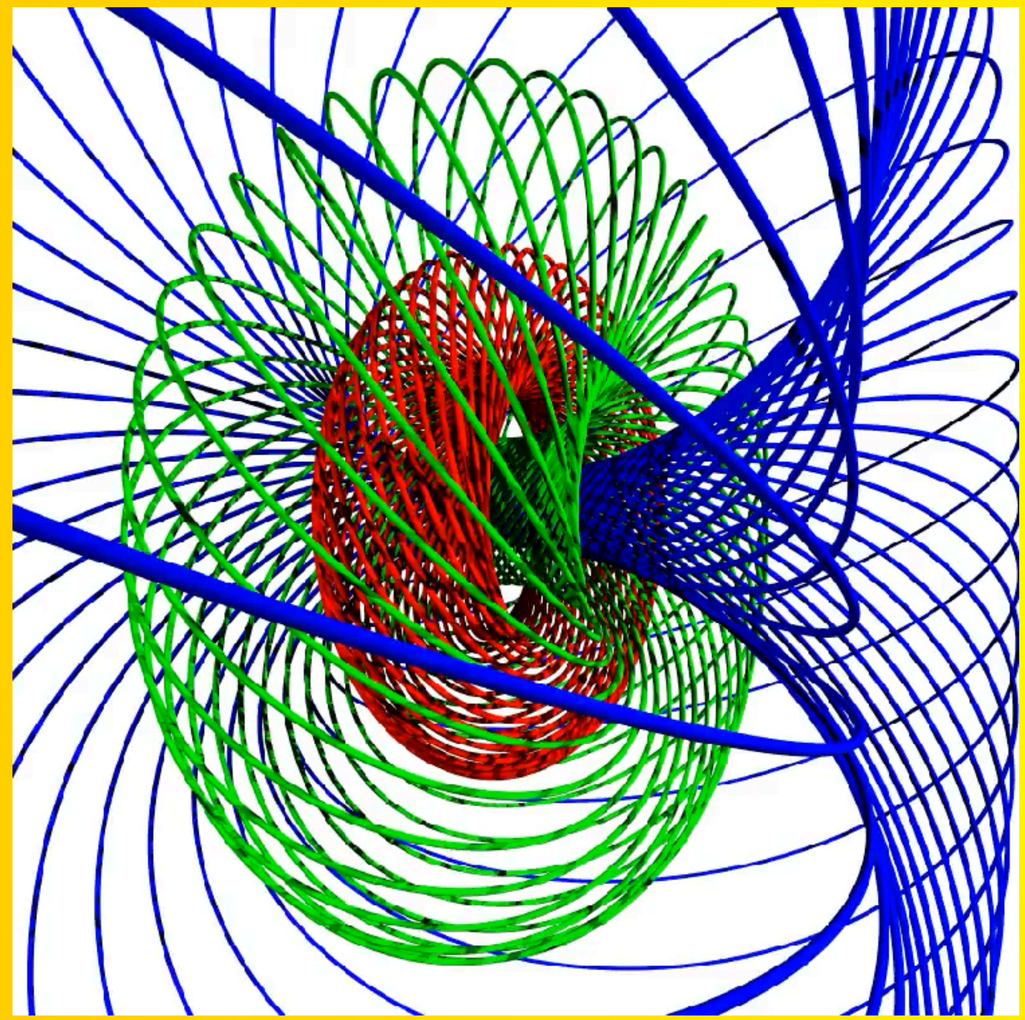
$$
\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta) \\
z &= z
\end{aligned}
$$

Spherical Coordinates



$$
\begin{aligned}
x &= \rho \sin(\phi)\cos(\theta) \\
y &= \rho \sin(\phi)\sin(\theta) \\
z &= \rho \cos(\phi)
\end{aligned}
$$

Works in any dimension



Parametrization of the sphere in 4D

$$
\begin{aligned}
x &= \rho \sin(\phi)\cos(\theta_1) \\
y &= \rho \sin(\phi)\sin(\theta_1) \\
x &= \rho \cos(\phi)\cos(\theta_2) \\
y &= \rho \cos(\phi)\sin(\theta_2)
\end{aligned}
$$

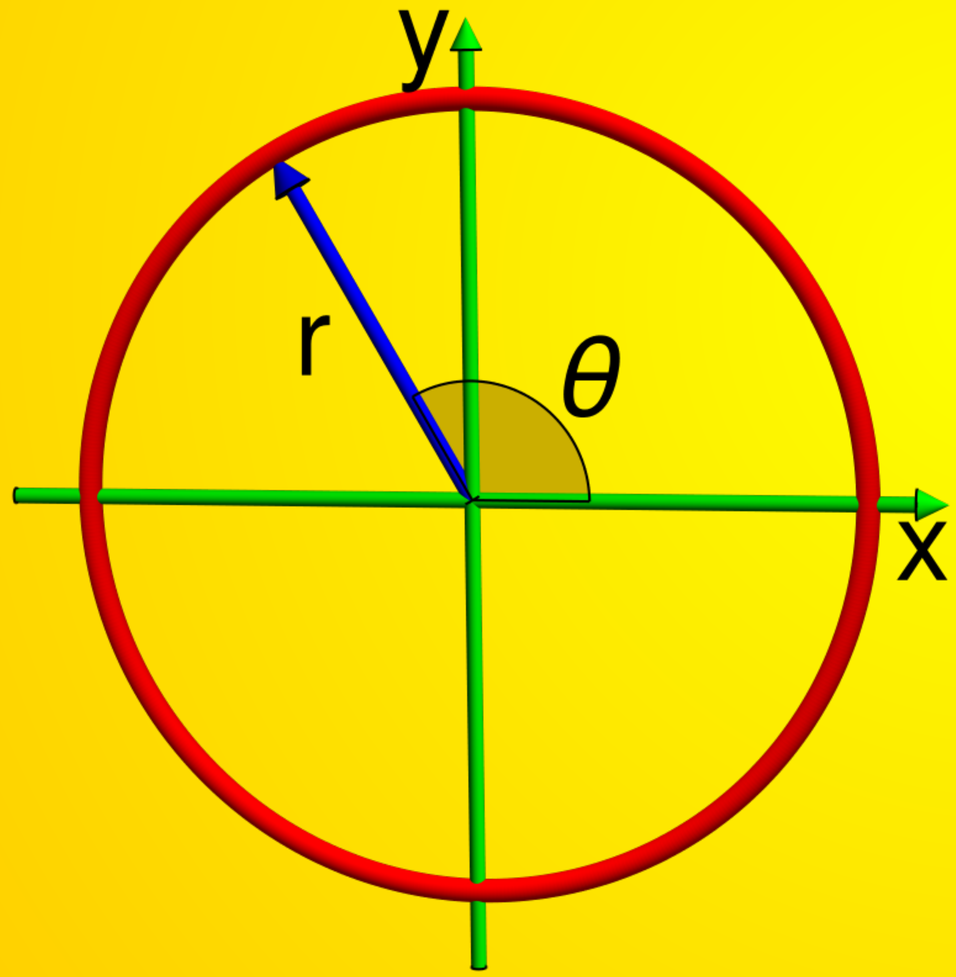
The Idea

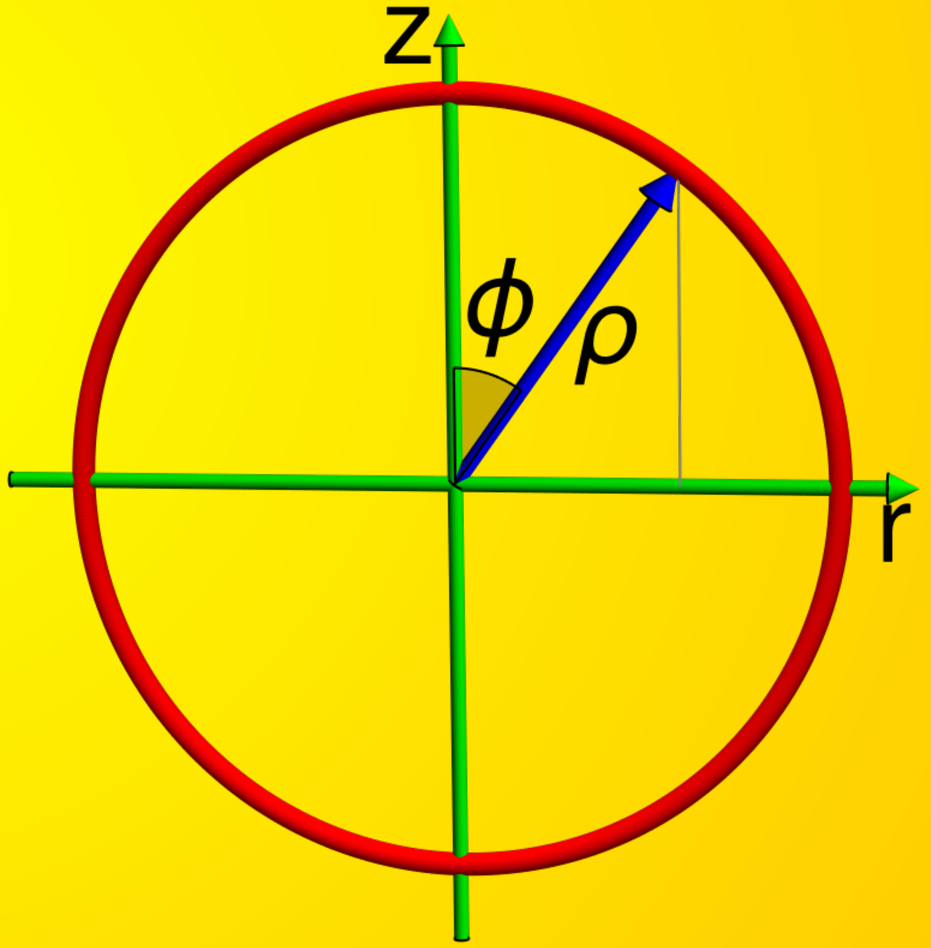
Leohard Euler
1707-1783

Refined

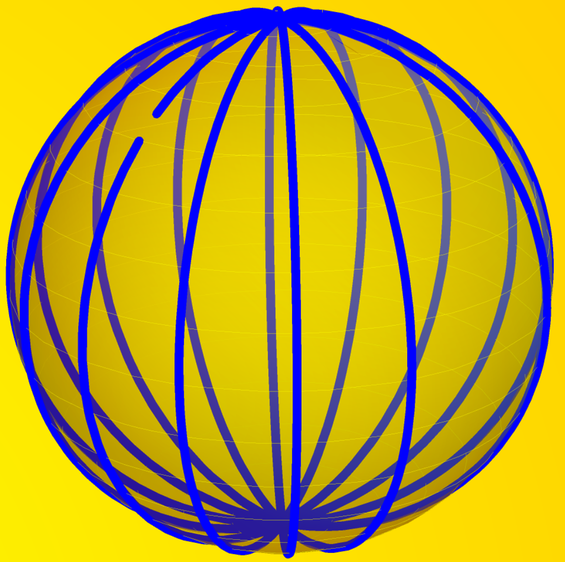
Fritz Zwicky
1898-1974

How to remember

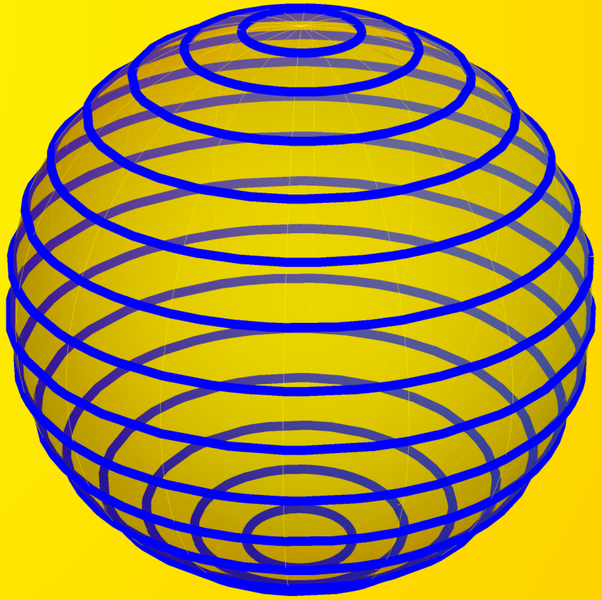




Longitude Lattitude



Longitude (West/East)



Latitude (North/South)

A

What is $\theta = 0$ on earth?

B

What is $\phi = 0$ on earth?

C

What is $\phi = \pi/2$ on earth?

D

What is $\phi = \pi$ on earth?

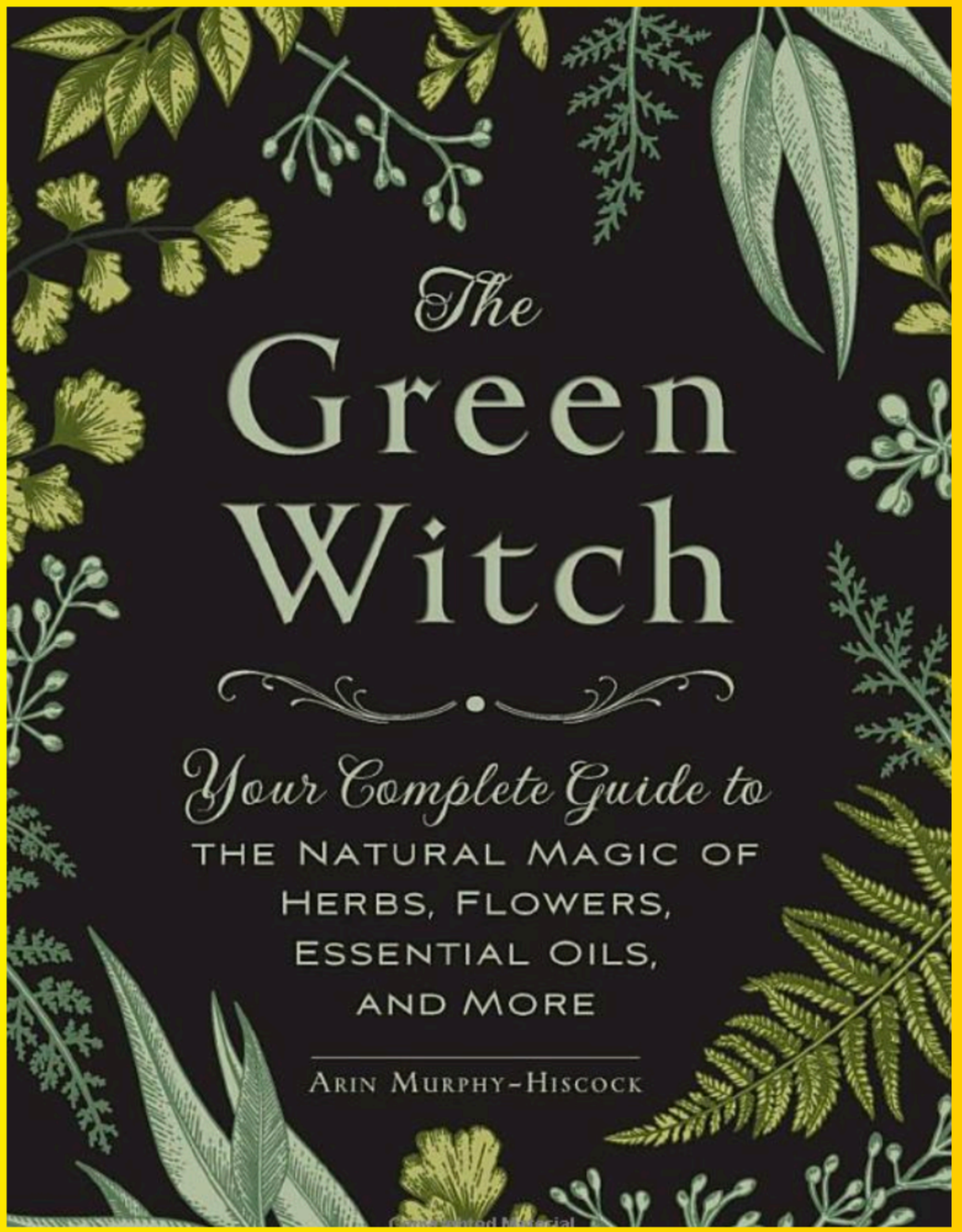
E

What is $\phi = \pi/6$ on earth?

Riddle

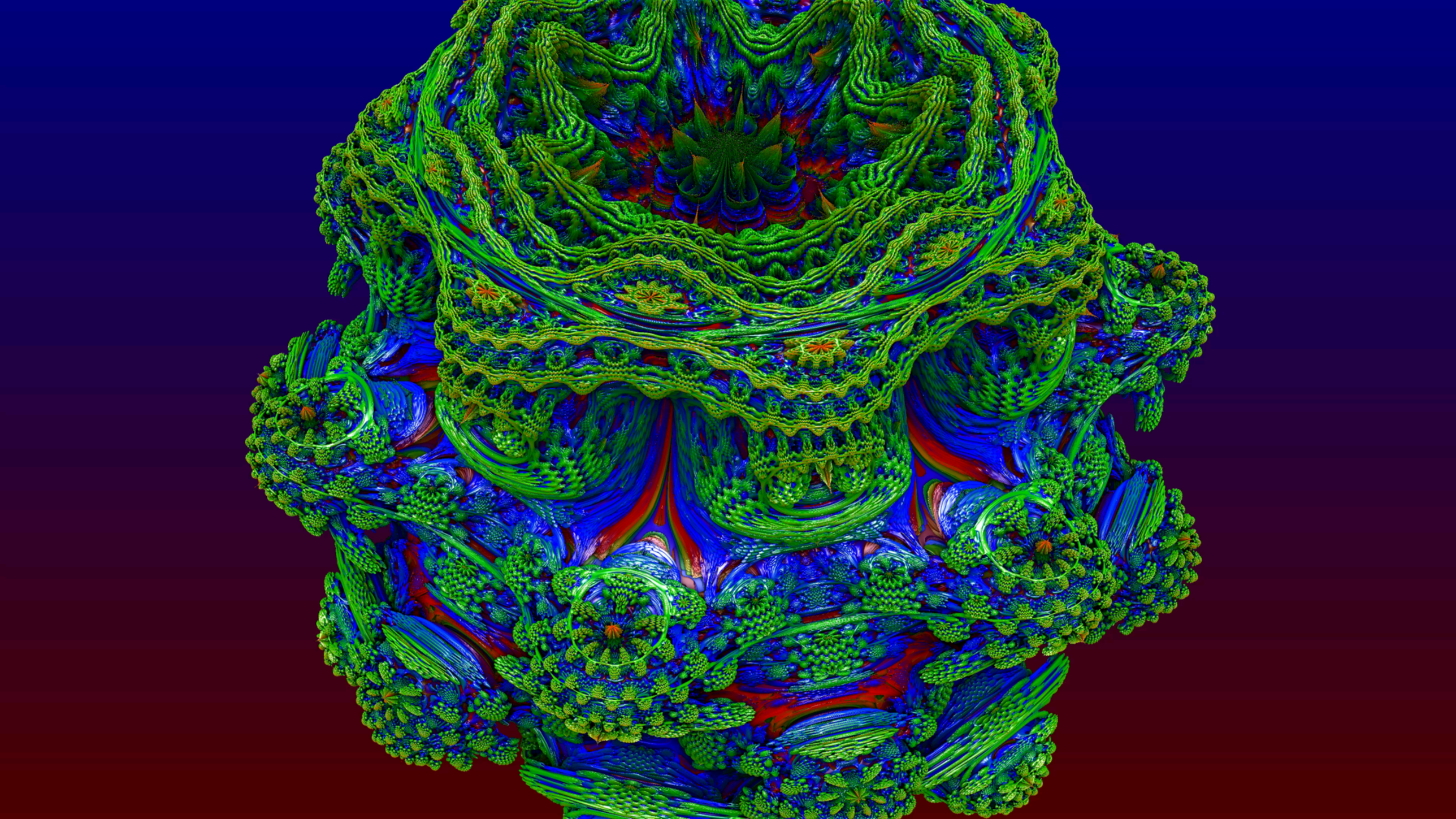
Riddle

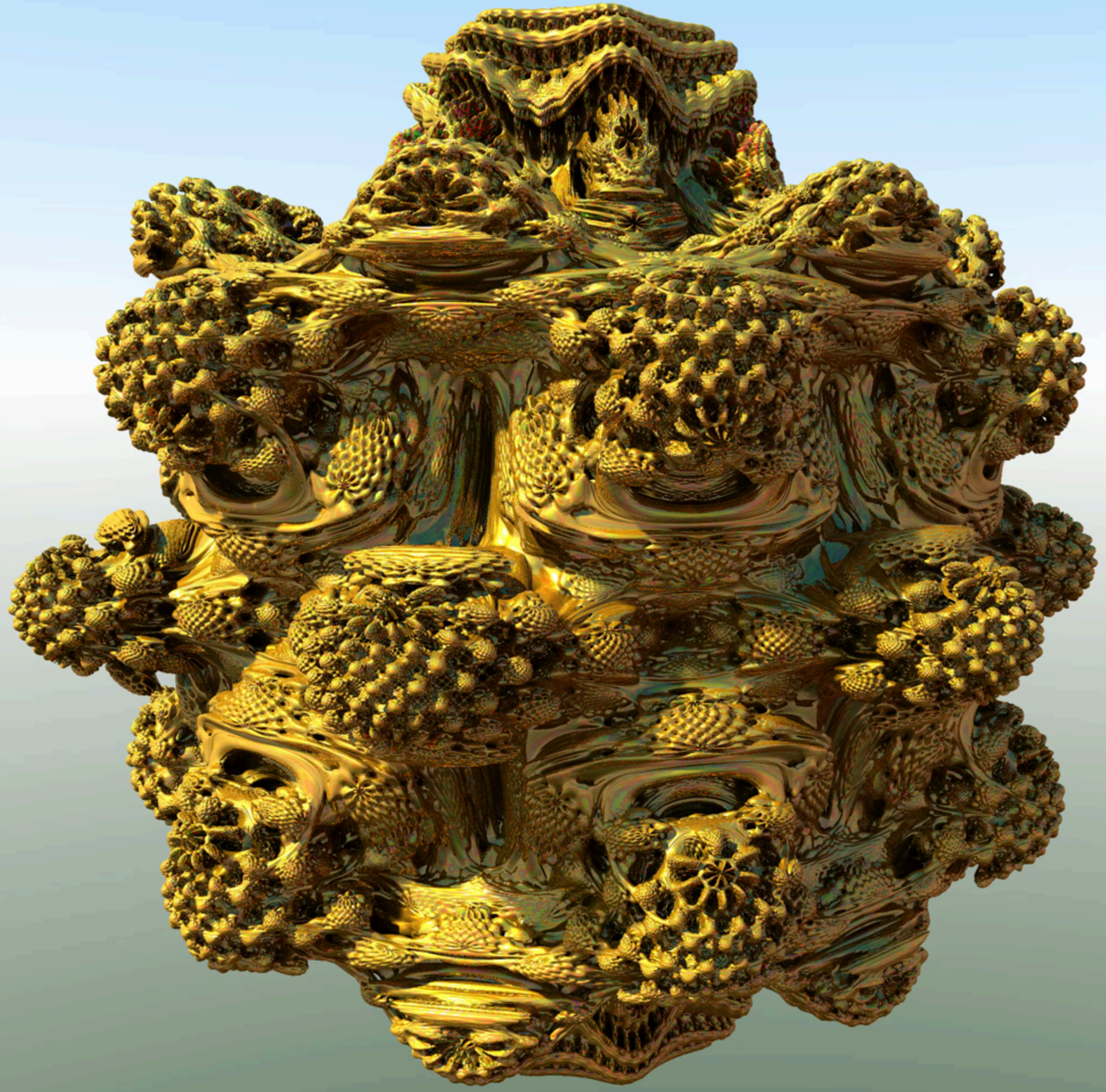
On which longitude does the Green Witch live?



Some Objects

Mandelbulb





Homework due Monday

Problem Set 3 - Cylindrical and Spherical Coordinates

You should be able to:

- Describe the geometric meaning of the polar coordinates r, θ; cylindrical coordinates r, θ, and z; and the spherical coordinates ρ, ϕ, and θ.
- Convert points and equations between Cartesian and polar coordinates in $\mathbb{R}^2$; and Cartesian, cylindrical, and spherical coordinates in $\mathbb{R}^3$.
- Sketch and describe curves, surfaces, and solids given in terms of Cartesian, polar, cylindrical or spherical coordinates.

Note: In Math 21a, we always use $r \geq 0$ and $\rho \geq 0$; not all sources use this convention, so be careful when you're looking at other sources! (The OpenStax textbook allows r and ρ to be negative.)

Also, physicists typically use a different naming convention than mathematicians; what we call r and θ, physicists usually call ρ and ϕ, and vice versa. If you develop a strong geometric understanding of what these coordinates represent, then you'll find that it's easy to switch to using different names if you ever need to.

Advice from former students. In past semesters, we've asked students to give study advice to future 21a students. Here's one piece of advice they gave; we'll share more on future problem sets.

"Don't be afraid to ask questions for clarification in class. Don't let yourself not understand things for extended periods of time because that just accumulates and that's no bueno."

1. If you haven't already, please talk with someone at the MQC or office hours, or get together with at least one other student outside of class sometime in the next week. As a reminder, we'll ask you to write a little about your experience doing this on Problem Set 6 (due Monday 9/19) and to share in class, so that you can hear from your peers about what different resources are like.

   (You don't need to submit anything now.)

2. Go to https://www.geogebra.org/m/zqywmtud; you should see 5 points labeled A, B, C, D, E on a sphere in $\mathbb{R}^3$. Suppose the sphere is described by $\rho = 2$.

   In each part, decide which of the 5 points is described by the given spherical coordinates; no explanation is necessary.

   (a) $\rho = 2$, $\theta = 0$, and $\phi = \frac{2\pi}{3}$ (b) $\rho = 2$, $\theta = \frac{2\pi}{3}$, and $\phi = 0$ (c) $\rho = 2$, $\theta = 4\pi$, and $\phi = \frac{2\pi}{3}$

3. (a) In each part, you're given a set of equations / inequalities describing an object in $\mathbb{R}^3$. Sketch the object, describe it in words, and say what its dimension is.

   i. $\rho = 7$ and $r = 2$

   ii. $\rho \leq 7$ and $r = 2$

   iii. $\phi = \frac{\pi}{2}$ and $0 \leq \theta \leq \frac{\pi}{4}$

   iv. $\phi = \frac{\pi}{4}$, $r = 2$, and $-\frac{\pi}{2} < \theta < \pi$

   v. $\rho \sin\phi = 2$

   (b) One of the objects in (a) was a 2-dimensional object with finite surface area. Which one? Find its surface area.

4. *The following problem is meant to help you practice flexible problem solving in $\mathbb{R}^3$; if you can visualize 2 spheres in $\mathbb{R}^3$, you should be able to come up with a good strategy for solving the problem!*

   Find the distance[1] between the sphere $(x-2)^2+(y+4)^2+(z-6)^2 = 25$ and the sphere $(x+7)^2+(y-3)^2+(z-5)^2 = 2$. Explain your strategy in words, and include a sketch to illustrate your strategy.

5. Next week, we'll start talking about integrals of functions of more than 1 variable. To prepare for this, it's important to really understand single-variable integrals! So, please watch this video, and write a summary of the key points. Also include any questions you have about what the video covers.

6. *At the end of every problem set, it's important to take 10 minutes to solidify what you've learned. Here, we'll explicitly ask you to do that, but even when we don't ask, you should make this type of reflection a part of your regular routine.*

   (a) Research shows that preparing to teach material leads to better learning than just trying to learn the material. So, imagine you're preparing to teach this material to your roommate. How would you answer the following questions?

   - How do you visualize what r and ρ represent?
   - θ and ϕ are both angles; what possible values can they have? Why?
   - How do you express x, y, z in terms of r, θ, z? How about in terms of ρ, ϕ, θ?
   - How do you express r and ρ in terms of x, y, z?

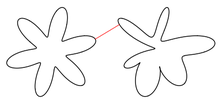
   Write out your responses.

   (b) *It's also important to think about how the new material you've learned relates to what you already know. This problem is an example of that.*

   In #3(a), do your answers agree with the rule of thumb about dimension that we've discussed? Explain briefly.

For next class, read *Active Calculus Multivariable* §10.2 - 10.3.

[1] The distance between two objects always means the *closest* distance; for example, the distance between the two blobs here is the length of the red line segment:



THE END

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