LINEAR EQUATIONS — Math 21b, O. Knill, 2018

SYSTEM OF LINEAR EQUATIONS. A collection of linear equations is called a **system of linear equations**. Here is an example:

$$\left|\begin{array}{ll} x+y+z & =1 \\ x+y & =2 \\ x+z & =3 \end{array}\right| .$$

This system consists of three equations for three **unknowns** x, y, z. **Linear** means that no nonlinear terms like $x^2, x^3, xy, yz^3, \sin(x)$ appear. Can you find the solution?

LINEAR EQUATION. A **linear equation** in n variables has the form $\boxed{a_1x_1 + a_2x_2 + \ldots + a_nx_n = a_0}$. Finitely many of such equations form a **system of linear equations**.

SOLVING BY ELIMINATION.
Eliminate variables. In the above example, we can write from the second equation $y = 2 - x$ and from the third equation $z = 3 - x$. If we plug this into the first equation we have $5 - x = 1$ or $x = 4$. Now the other equations can be fixed.

SOLVE BY SUITABLE SUBTRACTION.
Addition of equations. If we subtract the second equation from the first, we get $z = -1$. If we subtract the third equation from the first, we get $y = -2$.

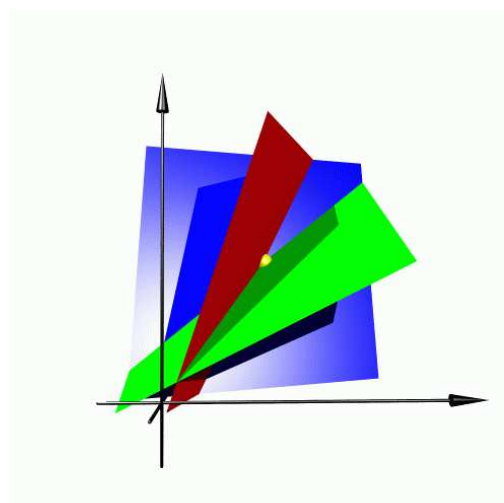
SOLVE BY COMPUTER.
Use the computer. In Mathematica:

$$\text{Solve}[\{x+y+z==1, x+y==2, x+z==3\}, \{x,y,z\}] .$$

But what did Mathematica do to solve this equation? We will look look at some algorithms. You can also type in the equations into a calculator. As for now, we don't recommend to do that as you need practice.

GEOMETRIC SOLUTION.

Each of the three equations represents a plane in three-dimensional space. Points on the first plane satisfy the first equation. The second plane is the solution set to the second equation. To satisfy the first two equations means to be on the intersection of these two planes which is here a line. To satisfy all three equations, we have to intersect the line with the plane representing the third equation which is a point.



LINES, PLANES, HYPERPLANES.
The set of points in the plane satisfying $ax + by = c$ form a **line**.
The set of points in space satisfying $ax + by + cd = d$ form a **plane**.
The set of points in n-dimensional space satisfying $a_1x_1 + ... + a_nx_n = a_0$ define a set called a **hyperplane**.

RIDDLES:

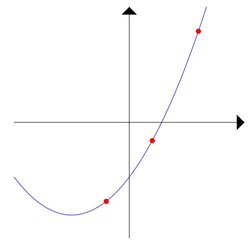
"25 kids have bicycles or tricycles. Together they count 60 wheels. How many have bicycles?"

Solution. With x bicycles and y tricycles, then $x + y = 25, 2x + 3y = 60$. The solution is $x = 15, y = 10$. One can get the solution by taking away $2 \cdot 25 = 50$ wheels from the 60 wheels. This counts the number of tricycles.

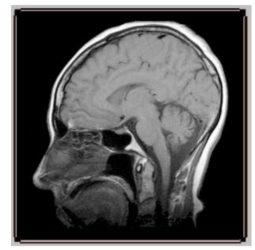
INTERPOLATION.

Find the equation of the parabola which passes through the points $P = (0,-1)$, $Q = (1,4)$ and $R = (2,13)$.

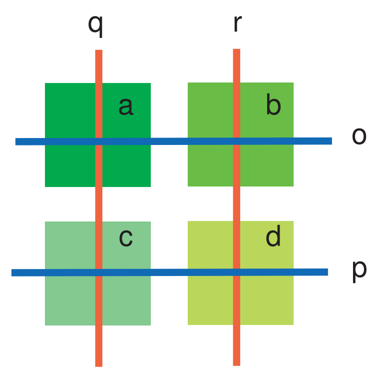
Solution. Assume the parabola consists of the set of points (x,y) which satisfy the equation $ax^2 + bx + c = y$. So, $c = -1, a + b + c = 4, 4a + 2b + c = 13$. Elimination of c gives $a + b = 5, 4a + 2b = 14$ so that $2b = 6$ and $b = 3, a = 2$. The parabola has the equation $2x^2 + 3x - 1 = 0$



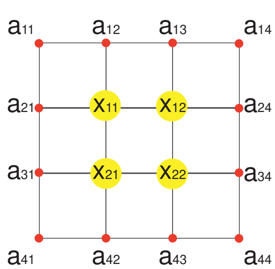
TOMOGRAPHY
Here is a toy example of a problem one has to solve for magnetic resonance imaging (MRI).



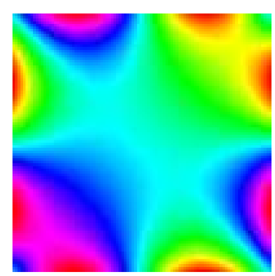
Assume we have 4 hydrogen atoms, whose nuclei are excited with energy intensity a,b,c,d. We measure the spin echo in 4 different directions. $3 = a + b$,$7 = c + d$,$5 = a + c$ and $5 = b + d$. What is a,b,c,d? Solution: $a = 2, b = 1, c = 3, d = 4$. However, also $a = 0, b = 3, c = 5, d = 2$ solves the problem. This system has not a unique solution even so there are 4 equations and 4 unknowns. A good introduction to MRI can be found online at (http://www.cis.rit.edu/htbooks/mri/inside.htm).



INCONSISTENT. $x - y = 4, y + z = 5, x + z = 6$ is a system with no solutions. It is called **inconsistent**.



EQUILIBRIUM. We model a drum by a fine net. The heights at each interior node needs the average the heights of the 4 neighboring nodes. The height at the boundary is fixed. With n^2 nodes in the interior, we have to solve a system of n^2 equations. For example, for $n = 2$ (see left), the $n^2 = 4$ equations are $4x_{11} = a_{21} + a_{12} + x_{21} + x_{12}$, $4x_{12} = x_{11} + x_{13} + x_{22} + x_{22}$, $4x_{21} = x_{31} + x_{11} + x_{22} + a_{43}$, $4x_{22} = x_{12} + x_{21} + a_{43} + a_{34}$. To the right, we see the solution to a problem with $n = 300$, where the computer had to solve a system with $90'000$ variables.



ON THE HISTORY. In 2000 BC the Babylonians studied problems which led to linear equations but more in a Diophantine setup, where integer solutions were asked for.
150-170: Jiuzhang Suanshu: Nine chapters of mathematical art. 300: Bakshali manuscript, linear equations 829: Al Khawarizmi: systematic way to solve linear and quadratic equations 1748 Cramers rule solution using determinants 1809 Gauss: Motion of heavenly bodies, Gauss-Jordan elimination and least square solutions

LINEAR OR NONLINEAR?
a) **The ideal gas law** $PV = nKT$ for the P,V,T, the pressure p, volume V and temperature T of a gas.
b) **Einsteins mass-energy equation** $E = mc^2$ relates rest mass m with the energy E of a body.

MATRIX FORMULATION.

Consider the system of linear equations like

$$\left| \begin{array}{ll} 3x - y - z & = 0 \\ -x + 2y - z & = 0 \\ -x - y + 3z & = 9 \end{array} \right|$$

It can be written as $A\vec{x} = \vec{b}$, where A is a **matrix** and $\vec{x}, \vec{b}$ are column vectors. The **coefficient matrix** of the above system is

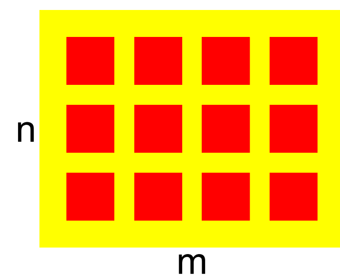
$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}.$$

The i'th entry b_i of Ax is the dot product of the i'th row with x.

The **augmented matrix** is the matrix in which an other column containing b has been added. It is often separated with a bar from the rest for clarity reasons.

$$B = \left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right].$$

MATRIX TERMINOLOGY.
A rectangular array of numbers is called a **matrix**. If the matrix has n **rows** and m **columns**, it is called a $n \times m$ matrix. A matrix with one column is a **column vector**, a matrix with one row is a **row vector**. The entries of a matrix are denoted by a_{ij}, where i is the row and j is the column. In the case of the linear equation above, the matrix A was a 3×3 square matrix and the augmented matrix B is a 3×4 matrix.



GAUSS-JORDAN ELIMINATION. Gauss-Jordan Elimination is a process, where successive subtraction of multiples of other rows or scaling brings the matrix into **reduced row echelon form**. The **elementary row operations** are

Swap two rows. | **S**cale a row | **S**ubtract a multiple of a row from an other.

The process transfers a given matrix A into a new matrix rref(A).

LEADING ONE. The first nonzero element in a row becomes a **leading one**, after the row has been rescaled so that this entry is 1. Gauss Jordan elimination aims to reach a leading one in each nonzero row.

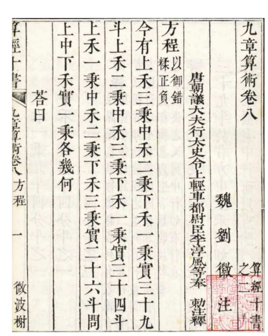
REDUCED ECHELON FORM. A matrix is in **reduced row echelon form**

1) in every nonzero row, the first nonzero entry is 1.
2) if a column has a leading 1, all other column entries are 0.
3) for a row with leading 1, every row above has a leading 1 to the left.

To memorize: Leaders like to be first, alone of their kind and have other leaders above to their left.

RANK. The number of leading 1 in rref(A) is called the rank of A.

HISTORY **Gauss Jordan elimination** appeared already in the Chinese manuscript "Jiuzhang Suanshu" ('Nine Chapters on the Mathematical art') a textbook from around 200 BC during the Han dynasty. The German geodesist Wilhelm **Jordan** (1842-1899) applied the Gauss-Jordan method to find squared errors in surveying. An other "Jordan", the French Mathematician Camille Jordan (1838-1922) worked on linear algebra topics also (Jordan form) and is often mistakenly credited with the Gauss-Jordan process. **Gauss** developed Gaussian elimination around 1800 and used it to solve least squares problems in celestial mechanics and later in geodesic computations. In 1809, Gauss published the book "Theory of Motion of the Heavenly Bodies" in which he used the method to solve astronomical problems like predicting the orbit of Ceres.



EXAMPLES. The reduced echelon form of the augmented matrix B will also be important to determine on how many solutions the linear system $Ax = b$ has.

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 2 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & -3 & -12 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\mathrm{rank}(A) = 3$, $\mathrm{rank}(B) = 3$.

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\mathrm{rank}(A) = 2$, $\mathrm{rank}(B) = 3$.

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathrm{rank}(A) = 2$, $\mathrm{rank}(B) = 2$.

PROBLEMS.

1) Find the rank of the following matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \end{bmatrix}.$$

2) Can you find a 2×5 matrix which is row reduced and has only 2 zero entries among the 10 matrix entries?

3) More challenging: what is the rank of the following matrix?

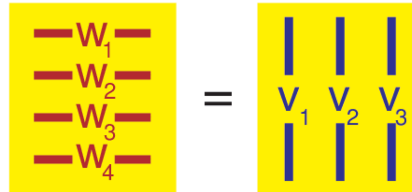
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \\ 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \\ 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 \\ 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}.$$

4) What are the possible ranks for a 7×11 matrix?

ON SOLUTIONS OF LINEAR EQUATIONS — Math 21b, O. Knill

A rectangular array of numbers is called a **matrix**.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$



A matrix with n **rows** and m **columns** is called a $n \times m$ matrix. A matrix with one column is a **column vector**. The entries of a matrix are denoted a_{ij}, where i is the row number and j is the column number. A system of linear equation defines a coefficient matrix A as well as an augmented matrix $[A|b]$.

ROW AND COLUMN PICTURE. Two interpretations

$$A\vec{x} = \begin{bmatrix} -\vec{w}_1- \\ -\vec{w}_2- \\ \cdots \\ -\vec{w}_n- \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \cdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b}\,.$$

"Row and Column at Harvard"

Row picture: each b_i is the dot product of a row vector $\vec{w}_i$ with $\vec{x}$.
Column picture: $\vec{b}$ is a sum of scaled column vectors $\vec{v}_j$.

EXAMPLE. The system of linear equations

$$\left| \begin{array}{ll} 3x - 4y - 5z & = 0 \\ -x + 2y - z & = 0 \\ -x - y + 3z & = 9 \end{array} \right|$$

is equivalent to $A\vec{x} = \vec{b}$, where A is a **coefficient matrix** and $\vec{x}$ and $\vec{b}$ are **vectors**.

$$A = \begin{bmatrix} 3 & -4 & -5 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}.$$

The **augmented matrix** (separators for clarity)

$$B = \left[\begin{array}{ccc|c} 3 & -4 & -5 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right].$$

In this case, the row vectors of A are
$\vec{w}_1 = \begin{bmatrix} 3 & -4 & -5 \end{bmatrix}$
$\vec{w}_2 = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$
$\vec{w}_3 = \begin{bmatrix} -1 & -1 & 3 \end{bmatrix}$
The column vectors are

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4 \\ -2 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$$

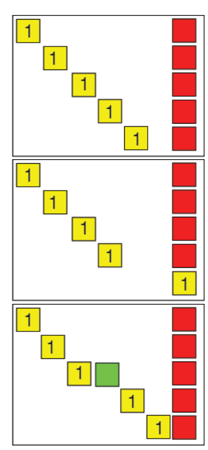
Row picture:

$$0 = b_1 = \begin{bmatrix} 3 & -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Column picture:

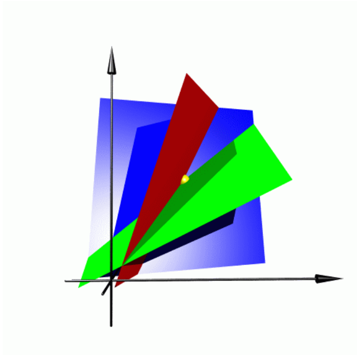
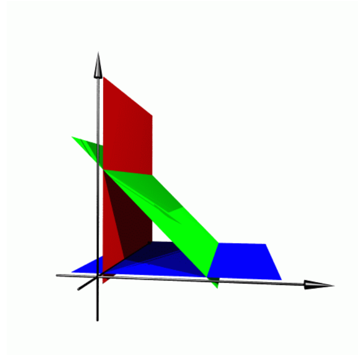
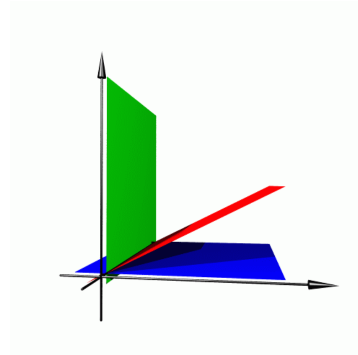
$$\begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

SOLUTIONS OF LINEAR EQUATIONS. A system $A\vec{x} = \vec{b}$ with n equations and m unknowns is defined by the $n \times m$ matrix A and the vector $\vec{b}$. The row reduced matrix rref(B) of the augmented matrix $B = [A|b]$ determines the number of solutions of the system $Ax = b$. The **rank** rank(A) of a matrix A is the number of leading ones in rref(A). There are three possibilities:
- **Consistent: Exactly one solution.** There is a leading 1 in each column of A but none in the last column of the augmented matrix B.
- **Inconsistent: No solutions.** There is a leading 1 in the last column of the augmented matrix B.
- **Consistent: Infinitely many solutions.** There are columns of A without leading 1.



If rank(A) = rank$(A|b) = m$, then there is **exactly 1 solution**.
If rank$(A) <$ rank$(A|b)$,there are **no solutions**.
If rank(A) = rank$(A|b) < m$: there are ∞ **solutions**.

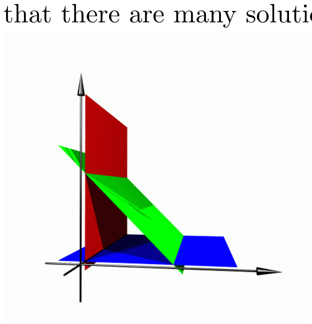
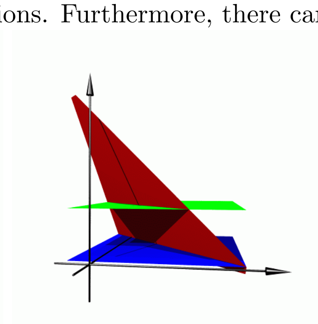
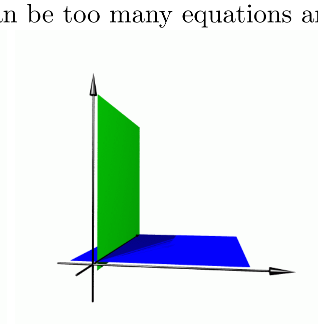
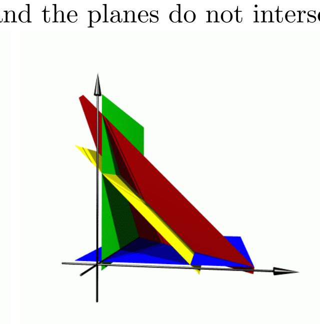
(exactly one solution) **(no solution)** **(infinitely many solutions)**

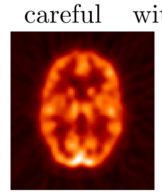
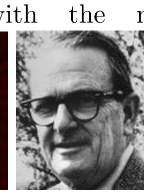
MURPHYS LAW.

"If anything can go wrong, it will go wrong".
"If you are feeling good, don't worry, you will get over it!"
"For Gauss-Jordan elimination, the error happens early in the process and get unnoticed."

MURPHY'S LAW IS TRUE. Two equations could contradict each other. Geometrically, the two planes do not intersect. This is possible if they are parallel. Even without two planes being parallel, it is possible that there is no intersection between all three of them. It is also possible that not enough equations are at hand or that there are many solutions. Furthermore, there can be too many equations and the planes do not intersect.

RELEVANCE OF EXCEPTIONAL CASES. There are important applications, where "unusual" situations happen: For example in medical tomography, systems of equations appear which are "ill posed". In this case one has to be careful with the method. The linear equations are then obtained from a method called the **Radon transform**. The task for finding a good method had led to a Nobel prize in Medicine 1979 for Allan Cormack. Cormack had sabbaticals at Harvard and probably has done part of his work on tomography here. Tomography helps to diagonse and cure.

MATRIX ALGEBRA I. Matrices can be added, subtracted if they have the same size:

$$A+B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}+b_{m2} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

They can also be scaled by a scalar λ:

$$\lambda A = \lambda \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

A system of linear equations can be written as $Ax = b$. If this system of equations has a unique solution, we write $x = A^{-1}b$, where A^{-1} is called the inverse matrix.

LINEAR TRANSFORMATIONS — Math 21b, O. Knill

TRANSFORMATIONS. A **transformation** T from X to Y is a rule, which assigns to every x in X an element $y = T(x)$ in Y. One calls X the **domain** and Y the **codomain** and T **maps** X to Y.

LINEAR TRANSFORMATION. A map T from $\mathbf{R}^m$ to $\mathbf{R}^n$ is called a **linear transformation** if there is a $n \times m$ matrix A such that $T(\vec{x}) = A\vec{x}$.

EXAMPLES.

- To the linear transformation $T(x, y) = (3x+4y, x+5y)$ belongs the matrix $\begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$. This transformation maps the plane onto itself.
- $T(x) = -33x$ is a linear transformation from the real line onto itself. The matrix is $A = [-33]$.
- To $T(\vec{x}) = \vec{y} \cdot \vec{x}$ from $\mathbf{R}^3$ to $\mathbf{R}$ belongs the matrix $A = \vec{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$. This 1×3 matrix is also called a **row vector**. If the codomain is the real axes, one calls the map also a **function**.
- $T(x) = x\vec{y}$ from $\mathbf{R}$ to $\mathbf{R}^3$. $A = \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is a 3×1 matrix which is also called a **column vector**. The map defines a line in space.
- $T(x, y, z) = (x, y)$ from $\mathbf{R}^3$ to $\mathbf{R}^2$, A is the 2×3 matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The map projects space onto a plane.
- To the map $T(x, y) = (x + y, x - y, 2x - 3y)$ belongs the 3×2 matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix}$. The image of the map is a plane in three dimensional space.
- If $T(\vec{x}) = \vec{x}$, then T is called the **identity transformation**.

PROPERTIES OF LINEAR TRANSFORMATIONS. $T(\vec{0}) = \vec{0}$ $\quad T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ $\quad T(\lambda\vec{x}) = \lambda T(\vec{x})$

In words: Linear transformations are compatible with addition and scalar multiplication and map 0 to 0. The later could be deduced from $\lambda = 0$ but its convenient to keep this property. It does not matter, whether we add two vectors before the transformation or add the transformed vectors.

ON LINEAR TRANSFORMATIONS. Linear transformations generalize the scaling transformation $x \mapsto ax$ in one dimensions.
They are important in

- geometry (i.e. rotations, dilations, projections or reflections)
- art (i.e. perspective, coordinate transformations),
- CAD applications (i.e. projections),
- physics (i.e. Lorentz transformations),
- dynamics (linearizations of general maps),
- compression (i.e. using Fourier transform or Wavelet transform),
- coding (many codes are linear codes),
- probability (i.e. Markov processes).
- linear equations (inversion is solving the equation)

COLUMN VECTORS. A linear transformation $T(x) = Ax$ with $A = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix}$ has the property that the column vector $\vec{v}_1, \vec{v}_i, \vec{v}_n$ are the images of the **standard vectors** $\vec{e}_1 = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$. $\vec{e}_i = \begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{bmatrix}$. $\vec{e}_n = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$.

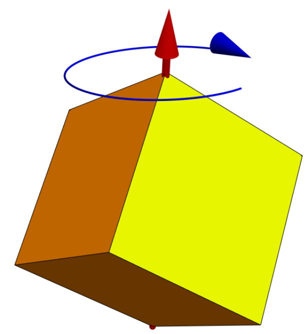
Here is one of the most important facts in the entire course:

In order to find the matrix of a linear transformation, look at the image of the standard vectors and use those to build the columns of the matrix. The vectors Ae_i "hanging" inside the matrix.

TWO GEOMETRIC PROBLEMS:

a) Find the matrix belonging to the linear transformation, which rotates a cube around the diagonal $(1, 1, 1)$ by 120 degrees $(2\pi/3)$.

b) Find the linear transformation, which reflects a vector at the line containing the vector $(1, 1, 1)$.



INVERSE OF A TRANSFORMATION. If there is a linear transformation S such that $S(T\vec{x}) = \vec{x}$ for every $\vec{x}$, then S is called the **inverse** of T. The inverse transformation is linear again!

SOLVING A LINEAR SYSTEM OF EQUATIONS. $A\vec{x} = \vec{b}$ means to invert the linear transformation $\vec{x} \mapsto A\vec{x}$. If the linear system has exactly one solution, then an inverse exists. We will write $\vec{x} = A^{-1}\vec{b}$ and see that the inverse of a linear transformation is again a linear transformation.

MATRIX CODE: We encrypt a message with the transformation $T(x, y) = (2x + 7y, 3x + 11y)$. This is given by a matrix

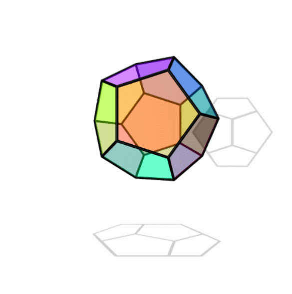
$$A = \begin{bmatrix} 2 & 7 \\ 3 & 11 \end{bmatrix}.$$

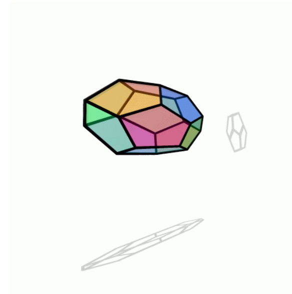
How do we get the matrix of the inverse transformation?

2x2 MATRIX. It is useful to decode the Matrix code in general. If $ax + by = X$ and $cx + dy = Y$, then $x = (dX - bY)/(ad - bc), y = (cX - aY)/(ad - bc)$. This is a linear transformation with matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the corresponding matrix is $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} /(ad - bc)$. You check that in the homework.

"Switch diagonally, negate the wings and scale with a cross".

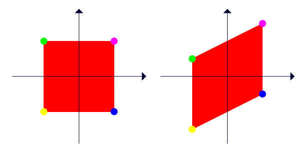
Linear Transformations can be used to rotate, reflect, twist or project. They are powerful tools to adjust geometric objects or describe motion in our physical world. They will be useful also in statistic, when doing data fitting.



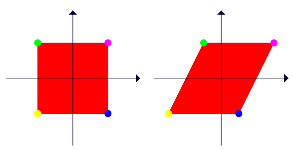


LINEAR TRANSFORMATIONS: we have seen that a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ is linear if there exists a matrix such that $T(x) = Ax$. We have also seen the properties: $T(\vec{0}) = \vec{0}$, $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(\lambda\vec{x}) = \lambda T(\vec{x})$ is a linear transformation.
Proof. Call $\vec{v}_i = T(\vec{e}_i)$ and define $S(\vec{x}) = A\vec{x}$. Then $S(\vec{e}_i) = T(\vec{e}_i)$. With $\vec{x} = x_1\vec{e}_1 + ... + x_n\vec{e}_n$, we have $T(\vec{x}) = T(x_1\vec{e}_1 + ... + x_n\vec{e}_n) = x_1\vec{v}_1 + ... + x_n\vec{v}_n$ as well as $S(\vec{x}) = A(x_1\vec{e}_1 + ... + x_n\vec{e}_n) = x_1\vec{v}_1 + ... + x_n\vec{v}_n$ proving $T(\vec{x}) = S(\vec{x}) = A\vec{x}$.

VERTICAL and HORIZONTAL SHEAR $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

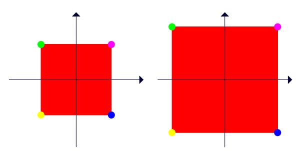


$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

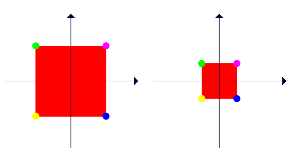


In general, shears are transformation in the plane with the property that there is a vector $\vec{w}$ such that $T(\vec{w}) = \vec{w}$ and $T(\vec{x}) - \vec{x}$ is a multiple of $\vec{w}$ for all $\vec{x}$. If $\vec{u}$ is orthogonal to $\vec{w}$, then $T(\vec{x}) = \vec{x} + (\vec{u}\cdot\vec{x})\vec{w}$.

SCALING = DILATION $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$



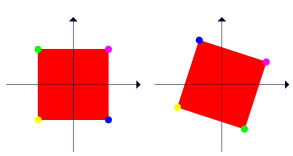
$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$



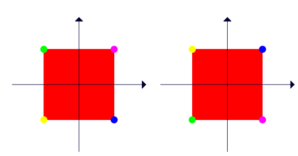
One can also look at transformations which scale x differently then y and where A is a diagonal matrix.

REFLECTION:

$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$



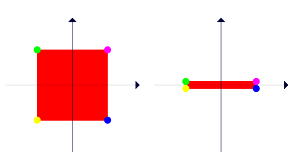
$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



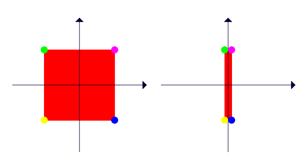
Any reflection at a line has the form of the matrix to the left. A reflection at a line containing a unit vector $\vec{u}$ is $T(\vec{x}) = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}$ with matrix $A = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$

PROJECTION:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



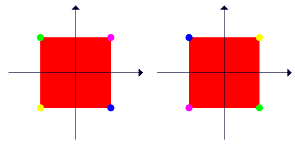
$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$



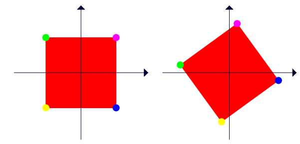
A projection onto a line containing unit vector $\vec{u}$ is $T(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$ with matrix $A = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix}$

ROTATION:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



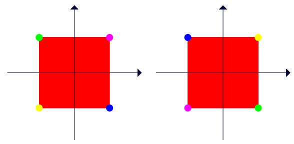
$$A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$



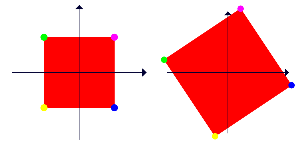
Any rotation has the form of the matrix to the right.

ROTATION-DILATION:

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$



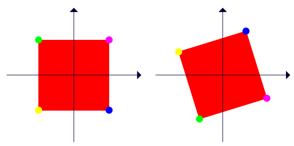
$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$



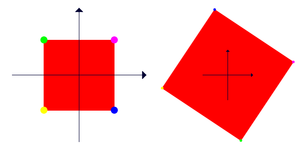
A rotation dilation is a composition of a rotation by angle $\arctan(y/x)$ and a dilation by a factor $\sqrt{x^2+y^2}$. If $z = x+iy$ and $w = a+ib$ and $T(x,y) = (X,Y)$, then $X+iY = zw$. So a rotation dilation is tied to the process of the multiplication with a complex number.

REFLECTION-DILATION:

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

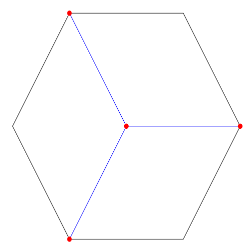


$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$



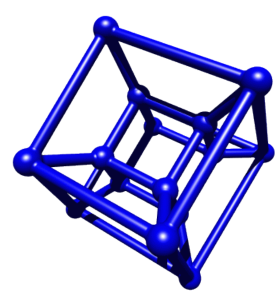
Besides rotation-dilations, also reflection dilations are important. We have seen already the structure of reflections. But there is a simpler way to write them down in general. Looking at the determinant helps to decide which case we have (see homework).

ROTATION IN SPACE. Rotations in space are defined by an axes of rotation and an angle. A rotation by $120°$ around a line containing $(0,0,0)$ and $(1,1,1)$ belongs to $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ which permutes $\vec{e}_1 \to \vec{e}_2 \to \vec{e}_3$.

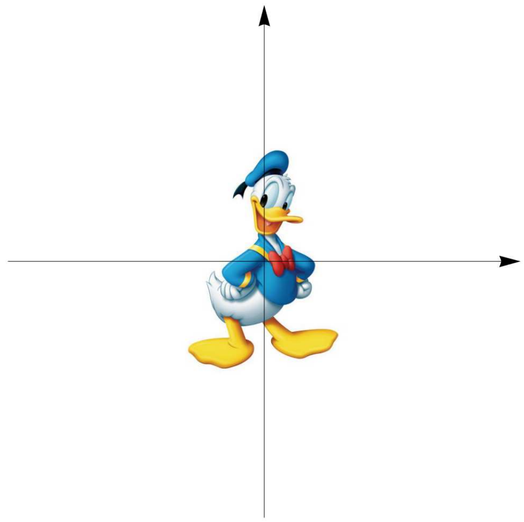


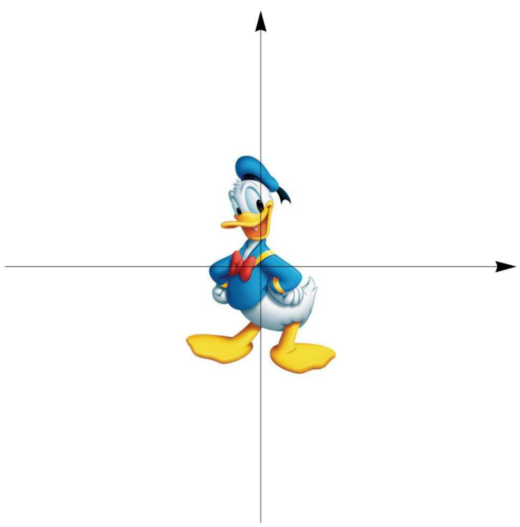
REFLECTION AT PLANE. To a reflection at the xy-plane belongs the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ as can be seen by looking at the images of $\vec{e}_i$. The picture to the right shows the textbook and reflections of it at two different mirrors.

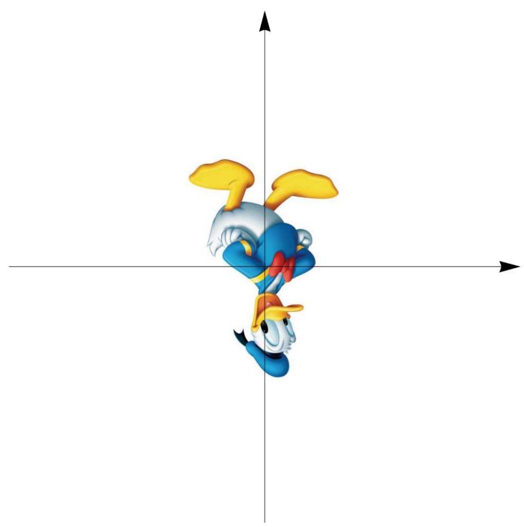
PROJECTION ONTO SPACE. To project a 4d-object into xyz-space, use for example the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The picture shows the projection of the four dimensional cube (tesseract, hypercube) with 16 edges $(\pm 1, \pm 1, \pm 1, \pm 1)$. The tesseract is the theme of the horror movie "hypercube".

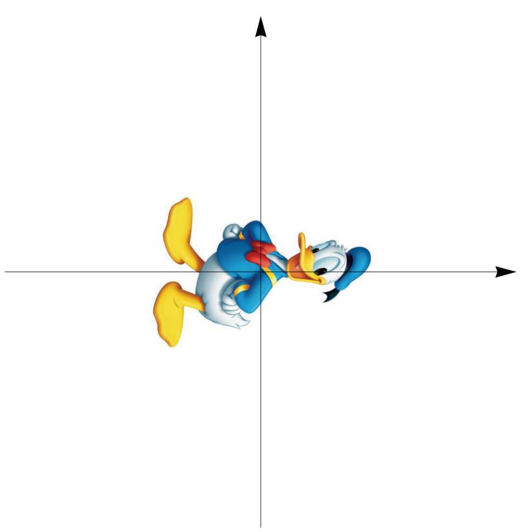


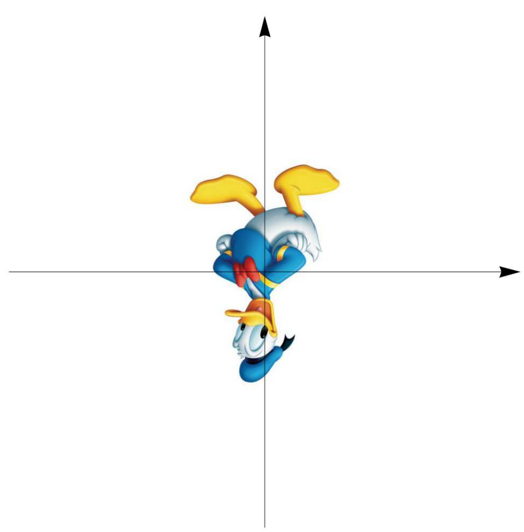
ICE: Can you find the matrix? **Math 21b, O. Knill**

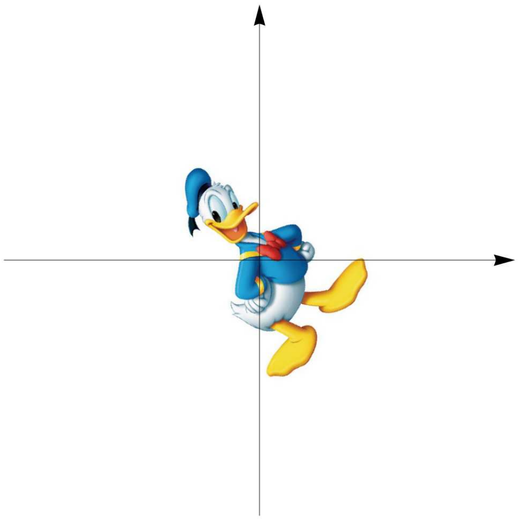
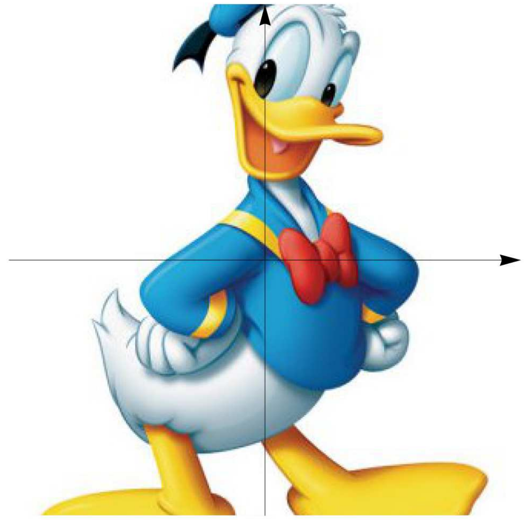
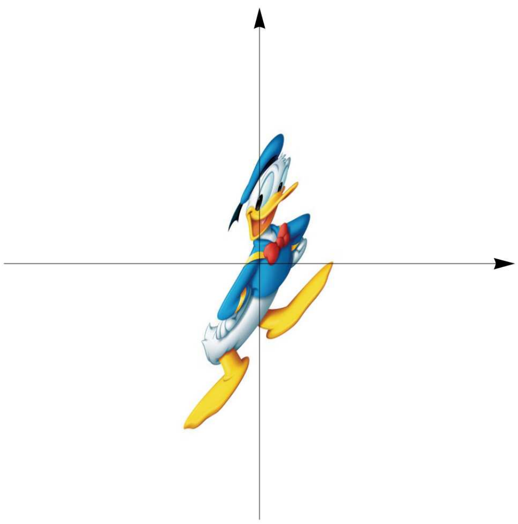
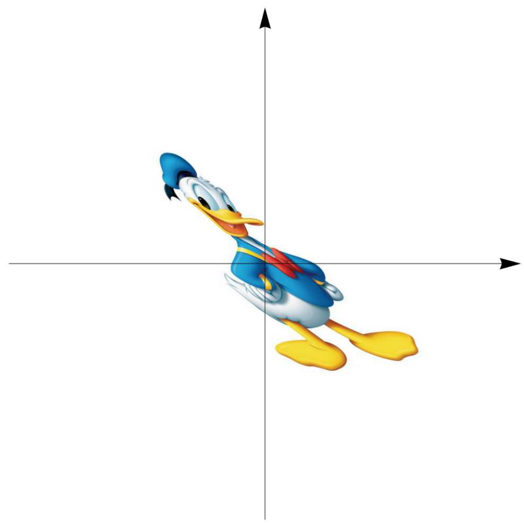
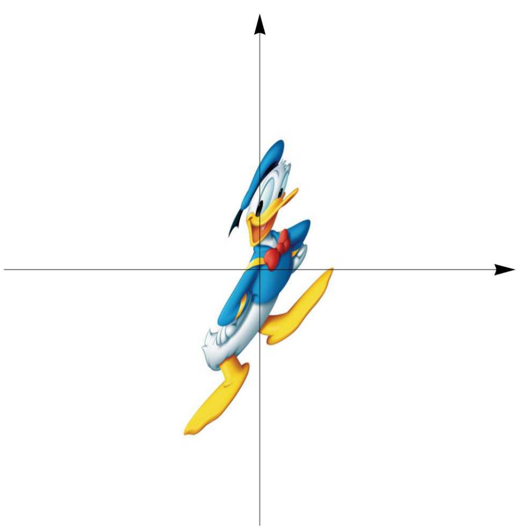
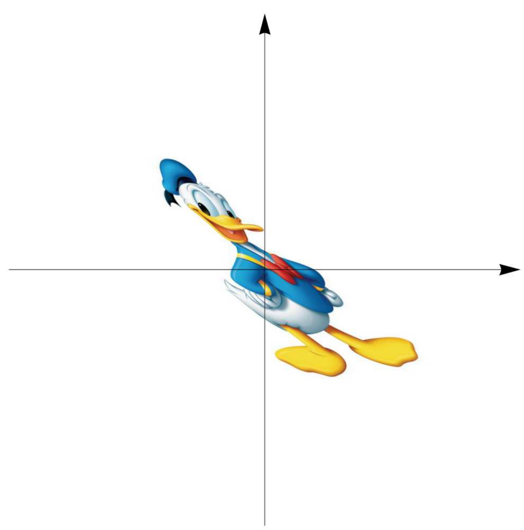










INVERTIBLE TRANSFORMATIONS. A map T from X to Y is called **invertible** if there exists for every $y \in Y$ a **unique** point $x \in X$ such that $T(x) = y$. $\Rightarrow$

EXAMPLES.
1) $T(x) = x^3$ is invertible from $X = \mathbf{R}$ to $X = Y$.
2) $T(x) = x^2$ is not invertible from $X = \mathbf{R}$ to $X = Y$.
3) $T(x, y) = (x^2 + 3x - y, x)$ is invertible from $X = \mathbf{R^2}$ to $Y = \mathbf{R}^2$.
4) $T(\vec{x}) = Ax$ linear and rref(A) has an empty row, then T is not invertible.
5) If $T(\vec{x}) = Ax$ is linear and rref$(A) = 1_n$, then T is invertible.

INVERSE OF LINEAR TRANSFORMATION. If A is a $n \times n$ matrix and $T : \vec{x} \mapsto Ax$ has an inverse S, then S is linear. The matrix A^{-1} belonging to $S = T^{-1}$ is called the **inverse matrix** of A.

First proof: check that S is linear using the characterization $S(\vec{a}+\vec{b}) = S(\vec{a}) + S(\vec{b}), S(\lambda\vec{a}) = \lambda S(\vec{a})$ of linearity.
Second proof: construct the inverse matrix using Gauss-Jordan elimination.

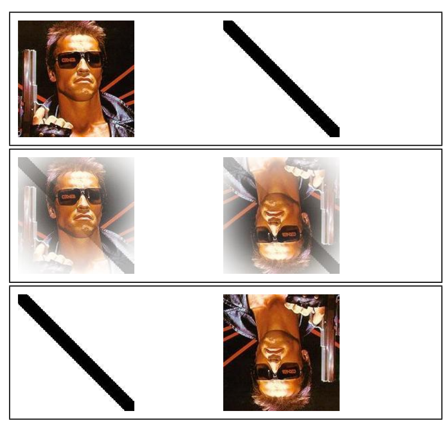
FINDING THE INVERSE. Let 1_n be the $n \times n$ identity matrix. Start with $[A|1_n]$ and perform Gauss-Jordan elimination. Then

$$\boxed{\text{rref}([A|1_n]) = \left[1_n|A^{-1}\right]}$$

Proof. The elimination process solves $A\vec{x} = \vec{e}_i$ simultaneously. This leads to solutions $\vec{v}_i$ which are the columns of the inverse matrix A^{-1} because $A^{-1}\vec{e}_i = \vec{v}_i$.

EXAMPLE. Find the inverse of $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \qquad \left[\; A \;|\; 1_2 \;\right]$$
$$\left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \qquad \left[\; .... \;|\; ... \;\right]$$
$$\left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1 \end{array}\right] \qquad \left[\; .... \;|\; ... \;\right]$$
$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1/2 & 1 \end{array}\right] \qquad \left[\; 1_2 \;|\; A^{-1} \;\right]$$



The inverse is $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1/2 & 1 \end{bmatrix}$.

THE INVERSE OF LINEAR MAPS $R^2 \mapsto R^2$:

If $ad-bc \neq 0$, the inverse of a linear transformation $\vec{x} \mapsto Ax$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by the matrix $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} /(ad - bc)$.

SHEAR:
$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\qquad$ $A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

DIAGONAL:
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

REFLECTION:
$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$ $\quad A^{-1} = A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$

ROTATION:
$A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(-\alpha) \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$

ROTATION-DILATION:
$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} a/r^2 & b/r^2 \\ -b/r^2 & a/r^2 \end{bmatrix}$, $r^2 = a^2 + b^2$

BOOST:
$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$

NONINVERTIBLE EXAMPLE. The projection $\vec{x} \mapsto A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a non-invertible transformation.

MORE ON SHEARS. The shears $T(x, y) = (x + ay, y)$ or $T(x, y) = (x, y + ax)$ in $\mathbf{R}^2$ can be generalized. A shear is a linear transformation which fixes some line L through the origin and which has the property that $T(\vec{x}) - \vec{x}$ is parallel to L for all $\vec{x}$. Shears are invertible.

PROBLEM. $T(x, y) = (3x/2 + y/2, y/2 - x/2)$ is a shear along a line L. Find L.
SOLUTION. Solve the system $T(x, y) = (x, y)$. You find that the vector $(1, -1)$ is preserved.

MORE ON PROJECTIONS. A linear map T with the property that $T(T(x)) = T(x)$ is a projection. Examples: $T(\vec{x}) = (\vec{y} \cdot \vec{x})\vec{y}$ is a projection onto a line spanned by a unit vector $\vec{y}$.

WHERE DO PROJECTIONS APPEAR? CAD: describe 3D objects using projections. A photo of an image is a projection. Compression algorithms like JPG or MPG or MP3 use projections where the high frequencies are cut away.

MORE ON ROTATIONS. A linear map T which preserves the angle between two vectors and the length of each vector is called a **rotation**. Rotations form an important class of transformations and will be treated later in more detail. In two dimensions, every rotation is of the form $x \mapsto A(x)$ with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$.
An example of a rotations in three dimensions are $\vec{x} \mapsto Ax$, with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. it is a rotation around the z axis.

MORE ON REFLECTIONS. Reflections are linear transformations different from the identity which are equal to their own inverse. Examples:
2D reflections at the origin: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, **2D reflections at a line** $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.
3D reflections at origin: $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. **3D reflections at a line** $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By the way: in any dimensions, to a reflection at the line containing the unit vector $\vec{u}$ belongs the matrix $[A]_{ij} = 2(u_i u_j) - [1_n]_{ij}$, because $[B]_{ij} = u_i u_j$ is the matrix belonging to the projection onto the line.
The reflection at a line containing the unit vector $\vec{u} = [u_1, u_2, u_3]$ is $A = \begin{bmatrix} u_1^2 - 1 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2^2 - 1 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3^2 - 1 \end{bmatrix}$.
3D reflection at a plane $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
Reflections are important symmetries in physics: T (time reflection), P (space reflection at a mirror), C (change of charge) are reflections. The composition TCP is a fundamental symmetry in nature.

MATRIX ALGEBRA — Math 21b, O. Knill

MATRIX PRODUCT. If A is a $n \times m$ matrix and A is a $m \times p$ matrix, then AB is defined as the $n \times p$ matrix with entries $(BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}$. It represents a linear transformation from $R^p \to R^n$ where first B is applied as a map from $R^p \to R^m$ and then the transformation A from $R^m \to R^n$.

EXAMPLE. If B is a 3×4 matrix, and A is a 4×2 matrix then BA is a 3×2 matrix.

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, BA = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 13 \\ 14 & 11 \\ 10 & 5 \end{bmatrix}.$$

EXAMPLE. If $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, then $A \cdot B = 4$ and $B \cdot A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

COMPOSING LINEAR TRANSFORMATIONS. If $T : \mathbf{R}^p \to \mathbf{R}^m, x \mapsto Bx$ and $S : \mathbf{R}^m \to \mathbf{R}^n, y \mapsto Ay$ are linear transformations, then their composition $S \circ T : x \mapsto A(B(x)) = ABx$ is a linear transformation from $\mathbf{R}^p$ to $\mathbf{R}^n$. The corresponding $n \times p$ matrix is the matrix product AB.

PROBLEM. Find the matrix which is a composition of a rotation around the x-axes by an angle $\pi/2$ followed by a rotation around the z-axes by an angle $\pi/2$.

SOLUTION. The first transformation has the property that $e_1 \to e_1, e_2 \to e_3, e_3 \to -e_2$, the second $e_1 \to e_2, e_2 \to -e_1, e_3 \to e_3$. If A is the matrix belonging to the first transformation and B the second, then BA is the matrix to the composition. The composition maps $e_1 \to -e_2 \to e_3 \to e_1$ is a rotation around a long diagonal. $B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$,

$BA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

EXAMPLE. A rotation dilation is the composition of a rotation by $\alpha = \arctan(b/a)$ and a dilation (=scale) by $r = \sqrt{a^2 + b^2}$.

MATRIX ALGEBRA. Note that $AB \neq BA$ in general and A^{-1} does not always exist, otherwise, the same rules apply as for numbers:

$A(BC) = (AB)C$, $AA^{-1} = A^{-1}A = 1_n$, $(AB)^{-1} = B^{-1}A^{-1}$, $A(B+C) = AB + AC$, $(B+C)A = BA + CA$.

INVERSE OF LINEAR TRANSFORMATION. If A is a $n \times n$ matrix and $T : \vec{x} \mapsto Ax$ has an inverse S, then S is linear. The matrix A^{-1} belonging to $S = T^{-1}$ is called the **inverse matrix** of A. We can construct the inverse matrix using Gauss-Jordan elimination.

FINDING THE INVERSE. Let 1_n be the $n \times n$ identity matrix. Start with $[A|1_n]$ and perform Gauss-Jordan elimination. Then

$$\boxed{\mathrm{rref}([A|1_n]) = [1_n|A^{-1}]}$$

Proof. The elimination process solves $A\vec{x} = \vec{e}_i$ simultaneously. This leads to solutions $\vec{v}_i$ which are the columns of the inverse matrix A^{-1} because $A^{-1}\vec{e}_i = \vec{v}_i$.

EXAMPLE. Find the inverse of $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$. It is $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1/2 & 1 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{c|c} A & 1_2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{c|c} .... & ... \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1 \end{array}\right] \qquad \left[\begin{array}{c|c} .... & ... \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1/2 & 1 \end{array}\right] \qquad \left[\begin{array}{c|c} 1_2 & A^{-1} \end{array}\right]$$

THE INVERSE OF LINEAR MAPS $R^2 \mapsto R^2$:

If $ad - bc \neq 0$, the inverse of a linear transformation $\vec{x} \mapsto Ax$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by the matrix $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} /(ad - bc)$.

SHEAR:

$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\qquad$ $A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

DIAGONAL:

$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\qquad$ $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

REFLECTION:

$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$ $\qquad$ $A^{-1} = A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$

ROTATION:

$A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(-\alpha) \end{bmatrix}$ $\qquad$ $A^{-1} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$

ROTATION-DILATION:

$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\qquad$ $A^{-1} = \begin{bmatrix} a/r^2 & b/r^2 \\ -b/r^2 & a/r^2 \end{bmatrix}$, $r^2 = a^2 + b^2$

REFLECTION-DILATION:

$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ $\qquad$ $A^{-1} = \begin{bmatrix} a/r^2 & b/r^2 \\ b/r^2 & -a/r^2 \end{bmatrix}$, $r^2 = a^2 + b^2$

IMAGE AND KERNEL

IMAGE. If $T : \mathbf{R}^m \to \mathbf{R}^n$ is a linear transformation, then $\{T(\vec{x}) \mid \vec{x} \in \mathbf{R}^m\ \}$ is called the **image** of T. If $T(\vec{x}) = A\vec{x}$, then the image of T is also called the image of A. We write $\mathrm{im}(A)$ or $\mathrm{im}(T)$.

EXAMPLES.
1) The map $T(x, y, z) = (x, y, 0)$ maps space into itself. It is linear because we can find a matrix A for which $T(\vec{x}) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The image of T is the xy-plane.
2) If $T(x, y) = (\cos(\phi)x - \sin(\phi)y, \sin(\phi)x + \cos(\phi)y)$ is a rotation in the plane, then the image of T is R^2.
3) If $T(x, y, z) = x + y + z$, then the image of T is $\mathbf{R}$.

SPAN. The **span** of vectors $\vec{v}_1, \dots, \vec{v}_k$ in $\mathbf{R}^n$ is the set of all combinations $c_1\vec{v}_1 + \dots c_k\vec{v}_k$, where c_i are real.

PROPERTIES.
The image of a linear transformation $\vec{x} \mapsto A\vec{x}$ is the span of the column vectors of A.
The image of a linear transformation contains 0 and is closed under addition and scalar multiplication.

KERNEL. If $T : \mathbf{R}^m \to \mathbf{R}^n$ is a linear transformation, then the set $\{x \mid T(x) = 0\ \}$ is called the **kernel** of T. If $T(\vec{x}) = A\vec{x}$, then the kernel of T is also called the kernel of A. We write $\ker(A)$ or $\ker(T)$.

EXAMPLES. (The same examples as above)
1) The kernel is the z-axes. Every vector $(0, 0, z)$ is mapped to 0.
2) The kernel consists only of the point $(0, 0)$.
3) The kernel consists of all vector (x, y, z) for which $x + y + z = 0$. The kernel is a plane.

PROPERTIES.
The kernel of a linear transformation contains 0 and is closed under addition and scalar multiplication.

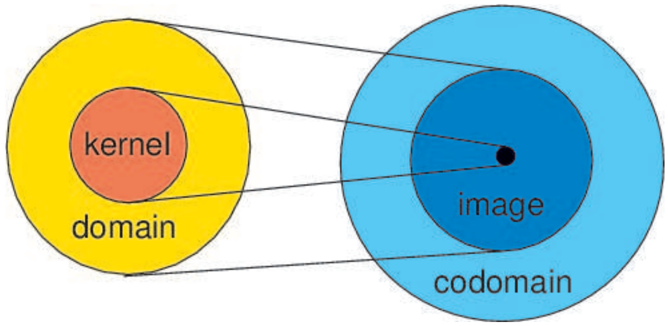
IMAGE AND KERNEL OF INVERTIBLE MAPS. A linear map $\vec{x} \mapsto A\vec{x}$, $\mathbf{R}^n \mapsto \mathbf{R}^n$ is invertible if and only if $\ker(A) = \{\vec{0}\}$ if and only if $\mathrm{im}(A) = \mathbf{R}^n$.

HOW DO WE COMPUTE THE IMAGE? The column vectors of A span the image. We will see later that the columns with leading ones alone span already the image.

EXAMPLES. (The same examples as above)
1) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ span the image.
2) $\begin{bmatrix} \cos(\phi) \\ -\sin(\phi) \end{bmatrix}$ and $\begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix}$ span the image.
3) The 1D vector $\begin{bmatrix} 1 \end{bmatrix}$ spans the image.

HOW DO WE COMPUTE THE KERNEL? Just solve the linear system of equations $A\vec{x} = \vec{0}$. Form $\mathrm{rref}(A)$. For every column without leading 1 we can introduce a **free variable** s_i. If $\vec{x}$ is the solution to $A\vec{x}_i = 0$, where all s_j are zero except $s_i = 1$, then $\vec{x} = \sum_j s_j\vec{x}_j$ is a general vector in the kernel.

EXAMPLE. Find the kernel of the linear map $\mathbf{R}^3 \to \mathbf{R}^4$, $\vec{x} \mapsto A\vec{x}$ with $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 5 \\ 3 & 9 & 1 \\ -2 & -6 & 0 \end{bmatrix}$. Gauss-Jordan elimination gives: $B = \mathrm{rref}(A) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We see one column without leading 1 (the second one). The equation $B\vec{x} = 0$ is equivalent to the system $x + 3y = 0, z = 0$. After fixing $z = 0$, can chose $y = t$ freely and obtain from the first equation $x = -3t$. Therefore, the kernel consists of vectors $t \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. In class, we compute more examples.



WHY DO WE LOOK AT THE KERNEL?

- It is useful to understand linear maps. To which degree are they non-invertible?
- It helps to describe solutions a linear equation $Ax = b$ has. If x is a solution and y is in the kernel of A, then also $A(x + y) = b$, so that $x + y$ solves the system also.

WHY DO WE LOOK AT THE IMAGE?

- A solution $Ax = b$ can be solved if and only if b is in the image of A.
- Knowing about the kernel and the image is useful in the similar way that it is useful to know about the domain and range of a general map and to understand the graph of the map.

The concept helps to understand applied topics like error correcting codes. where two matrices H, M with the property that $\ker(H) = \mathrm{im}(M)$ appear. The encoding $x \mapsto Mx$ is robust in the sense that adding an error e to the result $Mx \mapsto Mx + e$ can be corrected: $H(Mx + e) = He$ allows to find e and so Mx. This allows to recover $x = PMx$ with a projection P.

PROBLEM. Find $\ker(A)$ and $\mathrm{im}(A)$ for the 1×3 matrix $A = [5, 1, 4]$, a row vector.
ANSWER. $A \cdot \vec{x} = A\vec{x} = 5x + y + 4z = 0$ shows that the kernel is a plane with normal vector $[5, 1, 4]$ through the origin. The image is the codomain, which is $\mathbf{R}$.

PROBLEM. Let $v = [1, 1, 0]$ be a vector in space. Find $\ker(A)$ and image $\mathrm{im}(A)$ of $x \mapsto v \cdot x$.
ANSWER. (This is a more geometric reformulation of the problem before) The kernel consists of all vectors orthogonal to v, the image is the codomain, which is $\mathbf{R}$.

PROBLEM. Find $\ker(A)$ and $\mathrm{im}(A)$ of the linear map $x \mapsto v \times x$, the cross product with v.
ANSWER. The kernel consists of the line spanned by v, the image is the plane orthogonal to v.

PROBLEM Find $\ker(T)$ and $\mathrm{im}(T)$ if T is a composition of a rotation R by 90 degrees around the z-axes with with a projection onto the x-z plane.
ANSWER. The kernel of the projection is the y axes. The x axes is rotated into the y axes and therefore the kernel of T. The image is the x-z plane.

PROBLEM. Can the kernel of a square matrix A be trivial if $A^2 = \mathbf{0}$, where $\mathbf{0}$ is the matrix containing only 0?
ANSWER. No: if the kernel were trivial, then A were invertible and A^2 were invertible and be different from $\mathbf{0}$.

PROBLEM. Is it possible that a 3×3 matrix A satisfies $\ker(A) = \mathbf{R}^3$ without $A = \mathbf{0}$?
ANSWER. No, if $A \neq \mathbf{0}$, then A contains a nonzero entry and therefore, a column vector which is nonzero.

PROBLEM. What is the kernel and image of a projection onto the plane $\Sigma : x - y + 2z = 0$?
ANSWER. The kernel consists of all vectors orthogonal to Σ, the image is the plane Σ.

PROBLEM. Given two square matrices A, B and assume $AB = BA$. You know $\ker(A)$ and $\ker(B)$. What can you say about $\ker(AB)$?
ANSWER. $\ker(A)$ is contained in $\ker(BA)$. Similarly $\ker(B)$ is contained in $\ker(AB)$. Because $AB = BA$, the kernel of AB contains both $\ker(A)$ and $\ker(B)$. (It can be bigger as the example $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows.)

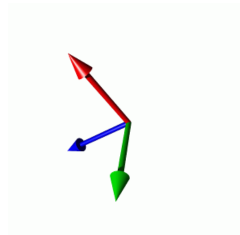
BASIS — Math 21b, O. Knill

LINEAR SPACE: A set of vectors which is called a **linear space** if three conditions hold: (a) $0 \in V$, (b) $\vec{v} + \vec{w} \in V$ if $\vec{v}, \vec{w} \in V$. (b) $\lambda \vec{v} \in V$ if $\vec{v}$ and λ is a real number.

WHICH OF THE FOLLOWING SETS ARE LINEAR SPACES?

a) The kernel of a matrix A.
b) The image of a matrix A.
c) The upper half plane.
d) The set $x^2 = y^2$.
e) the line $x + y = 0$.
f) The plane $x + y + z = 1$.
g) The unit circle.
h) The x-axes.

BASIS. A set of vectors $\vec{v}_1, \dots, \vec{v}_m$ is a **basis** of a linear subspace X of $\mathbf{R}^n$ if they are **linear independent** and if they **span** the space X. Linear independent means that there are no nontrivial **linear relations** $a_i\vec{v}_1 + \dots + a_m\vec{v}_m = 0$. Spanning the space means that very vector $\vec{v}$ can be written as a linear combination $\vec{v} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m$ of basis vectors.



EXAMPLE 1) The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis in the three dimensional space. If $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$, then $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$ and this representation is unique. We can find the coefficients by solving $A\vec{x} = \vec{v}$, where A has the v_i as column vectors. In our case, $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$ had the unique solution $x = 1, y = 2, z = 3$ leading to $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$.

EXAMPLE 2) Two nonzero vectors in the plane which are not parallel form a basis.
EXAMPLE 3) Four vectors in $\mathbf{R}^3$ are not a basis.
EXAMPLE 4) Two vectors in $\mathbf{R}^3$ never form a basis.
EXAMPLE 5) Three nonzero vectors in $\mathbf{R}^3$ which are not contained in a single plane form a basis in $\mathbf{R}^3$.
EXAMPLE 6) The columns of an invertible $n \times n$ matrix form a basis in $\mathbf{R}^n$.

FACT. If $\vec{v}_1, \dots, \vec{v}_n$ is a basis, then every vector $\vec{v}$ can be represented **uniquely** as a linear combination of the basis vectors: $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.

REASON. There is at least one representation because the vectors $\vec{v}_i$ span the space. If there were two different representations $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$ and $\vec{v} = b_1\vec{v}_1 + \dots + b_n\vec{v}_n$, then subtraction would lead to $0 = (a_1 - b_1)\vec{v}_1 + \dots + (a_n - b_n)\vec{v}_n$. Linear independence shows $a_1 - b_1 = a_2 - b_2 = ... = a_n - b_n = 0$.

FACT. If n vectors $\vec{v}_1, ..., \vec{v}_n$ span a space and $\vec{w}_1, ..., \vec{w}_m$ are linear independent, then $m \le n$.

REASON. This is intuitively clear in dimensions up to 3. You can not have 4 vectors in three dimensional space which are linearly independent. We will see this later.

A BASIS DEFINES AN INVERTIBLE MATRIX. The $n \times n$ matrix $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ is invertible if and only if $\vec{v}_1, \dots, \vec{v}_n$ define a basis in $\mathbf{R}^n$.

EXAMPLE. In example 1), the 3×3 matrix A is invertible.

FINDING A BASIS FOR THE KERNEL.
To find the kernel, solve $Ax = 0$. This means to bring the matrix A into the reduced row echelon form $\mathrm{rref}(A)$. For every non-leading entry in $\mathrm{rref}(A)$, we will get **free variables** t_i. Writing the system $Ax = 0$ with these free variables gives us an equation $\vec{x} = \sum_i t_i\vec{v}_i$. The vectors $\vec{v}_i$ form a basis of the kernel of A.

REMARK. The problem to find a basis for all vectors $\vec{w}_i$ which are orthogonal to a given set of vectors, is equivalent to the problem to find a basis for the kernel of the matrix which has the vectors $\vec{w}_i$ in its rows.

FINDING A BASIS FOR THE IMAGE. Bring the $m \times n$ matrix A into the form $\mathrm{rref}(A)$. Call a column **pivot**, if it contains a leading 1. The corresponding set of column vectors of the original matrix A form a basis for the image: they are linearly independent, are in the image. The pivot columns also span the image because if we remove the redundant columns, and $\vec{b}$ is in the image, we still can solve $A\vec{x} = b$.

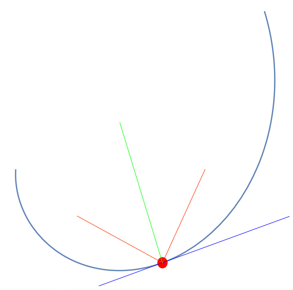
EXAMPLE. $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 4 & 5 \end{bmatrix}$. has two pivot columns, the first and second one. For $\vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, we can solve $A\vec{x} = \vec{b}$. We can also solve $B\vec{x} = \vec{b}$ with $B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

REMARK. The problem to find a basis of the subspace generated by $\vec{v}_1, \dots, \vec{v}_n$, is the problem to find a basis for the image of the matrix A with column vectors $\vec{v}_1, \dots, \vec{v}_n$.

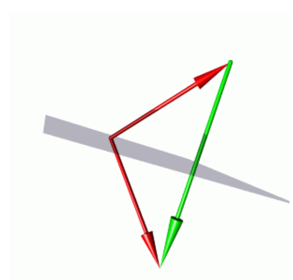
EXAMPLE. Let A be the matrix $A = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$. In reduced row echelon form is $B = \mathrm{rref}(A) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$. To determine a basis of the kernel we write $Bx = 0$ as a system of linear equations: $x + y = 0, z = 0$. The variable y is the free variable. With $y = t$, $x = -t$ is fixed. The linear system $\mathrm{rref}(A)x = 0$ is solved by $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. So, $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis of the kernel.

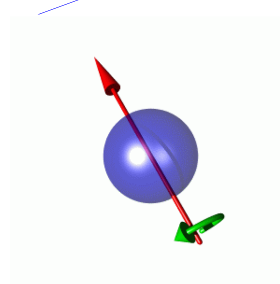
EXAMPLE. Because the first and third vectors in $\mathrm{rref}(A)$ are columns with leading 1's, the first and third columns $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ of A form a basis of the image of A.

WHY DO WE INTRODUCE BASIS VECTORS? Wouldn't it be just easier to always look at the standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$ only? The reason for the need of more general basis vectors is that they allow **a more flexible adaptation** to the situation. A person in Paris prefers a different set of basis vectors than a person in Boston. We will also see that in many applications, problems can be solved better.



For example, to describe the reflection of a ray at a plane or at a curve, it is preferable to use basis vectors which are tangent or orthogonal to the plane. When looking at a rotation, it is good to have one basis vector in the axis of rotation, the other two orthogonal to the axis. Choosing the right basis will be especially important when studying differential equations.





A PROBLEM. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find a basis for $\ker(A)$ and $\mathrm{im}(A)$.

SOLUTION. From $\mathrm{rref}(A) = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$. We see that there are two leading 1 and therefore one free variable (for the third column). Write $x - s = 0, y + 2s = 0, z = s$ so that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, see that $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is in the kernel. The two column vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ of A form a basis of the image because the first and third column are pivot columns.

DIMENSION — Math 21b, O. Knill

REVIEW LINEAR SPACES.. $X \subset R^n$ is a **linear space** if $0 \in X$ and if X is closed under addition and scalar multiplication. Examples are $\mathbf{R}^n, X = \ker(A), X = \mathrm{im}(A)$, or the row space of a matrix. In order to describe linear spaces, we introduced the notion of a basis:

REVIEW BASIS. $\mathcal{B} = \{v_1, \dots, v_n\} \subset X$
$\mathcal{B}$ **linear independent**: $c_1v_1 + ... + c_nv_n = 0$ implies $c_1 = ... = c_n = 0$.
$\mathcal{B}$ **span** X: $v \in X$ then $v = a_1v_1 + ... + a_nv_n$.
$\mathcal{B}$ **basis**: a set of vectors which both linear independent and span.

BASIS: ENOUGH BUT NOT TOO MUCH. The spanning condition for a basis assures that there are **enough** vectors to represent any other vector, the linear independence condition assures that there are **not too many** vectors. A basis is, where J.Lo meets A.Hi: Left: J.Lopez in **"Enough"**, right "The man who new **too much**" by A.Hitchcock

DIMENSION. The number of elements in a basis of X is independent of the choice of the basis. This works because if q vectors span X and p other vectors are independent then $q \geq p$ (see lemma) Applying this twice to two different basis with q or p elements shows $p = q$. The number of basis elements is called the **dimension** of X.

UNIQUE REPRESENTATION. $v_1, \dots, v_n \in X$ **basis** $\Rightarrow$ every $v \in X$ can be written uniquely as a sum $v = a_1v_1 + \dots + a_nv_n$.

EXAMPLES. The dimension of $\{0\}$ is zero. The dimension of a line 1. The dimension of a plane is 2, the dimension of three dimensional space is 3. The dimension is independent on where the space is embedded in. For example: a line in the plane and a line in space have dimension 1.

REVIEW: KERNEL AND IMAGE. We can construct a basis of the kernel and image of a linear transformation $T(x) = Ax$ by forming $B = \mathrm{rref}A$. The set of Pivot columns in A form a basis of the image of T, a basis for the kernel is obtained by solving $Bx = 0$ and introducing free variables for each non-pivot column.

PROBLEM. Find a basis of the span of the column vectors of A

$$A = \begin{bmatrix} 1 & 11 & 111 & 11 & 1 \\ 11 & 111 & 1111 & 111 & 11 \\ 111 & 1111 & 11111 & 1111 & 111 \end{bmatrix}.$$

Find also a basis of the **row space** the span of the row vectors.

SOLUTION. In order to find a basis of the column space, we row reduce the matrix A and identify the leading 1: we have

$$\mathrm{rref}(A) = \begin{bmatrix} \boxed{1} & 0 & -10 & 0 & 1 \\ 0 & \boxed{1} & 11 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because the first two columns have leading $\boxed{1}$, the first two columns of A span the image of A, the column space. The basis is $\{ \begin{bmatrix} 1 \\ 11 \\ 111 \end{bmatrix}, \begin{bmatrix} 11 \\ 111 \\ 1111 \end{bmatrix} \}$.
Now produce a matrix B which contains the rows of A as columns

$$B = \begin{bmatrix} 1 & 11 & 111 \\ 11 & 111 & 1111 \\ 111 & 1111 & 11111 \\ 11 & 111 & 1111 \\ 1 & 11 & 111 \end{bmatrix}$$

and row reduce it to

$$\mathrm{rref}(B) = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The first two columns of A span the image of B. $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 11 \\ 111 \\ 11 \\ 1 \end{bmatrix} \begin{bmatrix} 11 \\ 111 \\ 1111 \\ 111 \\ 11 \end{bmatrix} \}$.

Mathematicians call a fact a "lemma" if it is used to prove a theorem and if does not deserve the be honored by the name "theorem":

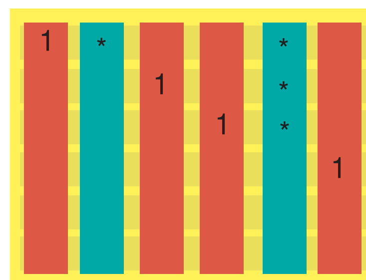
LEMMA. If q vectors $v_1, ..., v_q$ span X and $w_1, \dots, w_p$ are linearly independent in X, then $q \geq p$.

REASON. Assume $q < p$. Because v_i span, each vector w_i can be written as $w_i = \sum_{j=1}^{q} a_{ij} v_j$. Now row reduce the augmented $(p \times (q+n))$-matrix $\left| \begin{array}{ccccc} a_{11} & \dots & a_{1q} & | & w_1^T \\ \dots & \dots & \dots & | & \dots \\ a_{p1} & \dots & a_{pq} & | & w_p^T \end{array} \right|$, where v_i^T is the vector v_i written as a row vector. Each row of A of this matrix contains some nonzero entry. We end up with a matrix, which contains a last row $\left| \begin{array}{ccccc} 0 & \dots & 0 & | & b_1 w_1^T + ... + b_q w_q^T \end{array} \right|$ showing that $b_1 w_1^T + \cdots + b_q w_q^T = 0$. Not all b_j are zero because we had to eliminate some nonzero entries in the last row of A. This nontrivial relation of w_i contradicts linear independence. The assumption $q < p$ can not be true.

THEOREM. Given a basis $\mathcal{A} = \{v_1, ..., v_n\}$ and a basis $\mathcal{B} = \{w_1, ..., .w_m\}$ of X, then $m = n$.

PROOF. Because $\mathcal{A}$ spans X and $\mathcal{B}$ is linearly independent, we know that $n \leq m$. Because $\mathcal{B}$ spans X and $\mathcal{A}$ is linearly independent also $m \leq n$ holds. Together, $n \leq m$ and $m \leq n$ implies $n = m$.

DIMENSION OF THE KERNEL. The number of columns in rref(A) without leading 1's is the **dimension of the kernel** dim(ker(A)): we can introduce a parameter for each such column when solving $Ax = 0$ using Gauss-Jordan elimination. The dimension of the kernel of A is the number of "free variables" of A.



DIMENSION OF THE IMAGE. The number of **leading 1** in rref(A), the rank of A is the **dimension of the image** dim(im(A)) because every such leading 1 produces a different column vector (called **pivot column vectors**) and these column vectors are linearly independent.

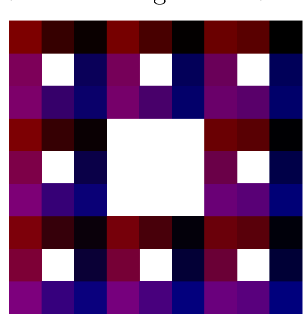
RANK-NULLITY THEOREM Let $A : \mathbf{R}^n \to \mathbf{R}^m$ be a linear map. Then

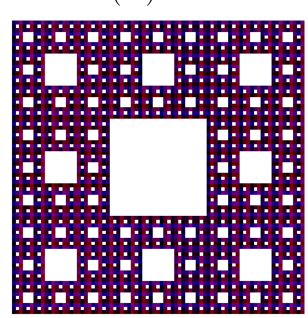
$$\boxed{\dim(\ker(A)) + \dim(\mathrm{im}(A)) = n}$$

This result is sometimes also called the **fundamental theorem of linear algebra**.
SPECIAL CASE: If A is an invertible $n \times n$ matrix, then the dimension of the image is n and that the dim(ker)$(A) = 0$.

PROOF OF THE DIMENSION FORMULA. There are n columns. dim(ker(A)) is the number of columns without leading 1, dim(im(A)) is the number of columns with leading 1.

FRACTAL DIMENSION. (just for fun) Mathematicians study objects with non-integer dimension since the early 20'th century. About 40 years ago mathematicians started to generate fracals. A fractal is an object with fractional dimension. This must first be defined: the **s-volume of accuracy** r of a bounded set X in $\mathbf{R}^n$ is the infimum of all $h_{s,r}(X) = \sum_{U_j} |U_j|^s$, where U_j are cubes of length $\leq r$ covering X and $|U_j|$ is the length of U_j. The **s-volume** is then defined as the limit $h_s(X)$ of $h_s(X) = h_{s,r}(X)$ when $r \to 0$. The **dimension** is the limiting value s, where $h_s(X)$ jumps from 0 to ∞. Examples:
1) A smooth curve X of length 1 in the plane can be covered with n squares U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^{n} (1/n)^s = n(1/n)^s$. If $s < 1$, this converges, if $s > 1$ it diverges for $n \to \infty$. So $\dim(X) = 1$.
2) A square X in space of area 1 can be covered with n^2 cubes U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^{n^2} (1/n)^s = n^2(1/n)^s$ which converges to 0 for $s < 2$ and diverges for $s > 2$ so that $\dim(X) = 2$.
3) The **Shirpinski carpet** is constructed recursively by dividing a square in 9 equal squares and cutting away the middle one, repeating this procedure with each of the squares etc. At the k'th step, we need 8^k squares of length $1/3^k$ to cover the carpet. The $s-$volume $h_{s,1/3^k}(X)$ of accuracy $1/3^k$ is $8^k (1/3^k)^s = 8^k/3^{ks}$, which goes to 0 for $k \to \infty$ if $3^{ks} < 8^k$ or $s < d = \log(8)/\log(3)$ and diverges if $s > d$. The dimension is $d = \log(8)/\log(3) = 1.893$..





$\mathcal{B}$-COORDINATES. A basis $\mathcal{B} = \{v_1, \dots v_n\}$ defines the matrix $S = \begin{bmatrix} | & \dots & | \\ v_1 & \dots & v_n \\ | & \dots & | \end{bmatrix}$. It is invertible. If $x = c_1 v_1 + \cdots + c_n v_n$, then c_i are called the **$\mathcal{B}$-coordinates** of v. We write $c = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}$. If $x = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$, we have $x = Sc$.

$\mathcal{B}$-coordinates of x are obtained by applying S^{-1} to the coordinates of the standard basis:

$$\boxed{c = S^{-1}(x)}$$

EXAMPLE. If $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, then $S = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. A vector $v = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ has the coordinates

$$S^{-1}v = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Indeed, as we can check, $-3v_1 + 3v_2 = v$.

EXAMPLE. Let V be the plane $x + y - z = 1$. Find a basis, in which every vector in the plane has the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$. SOLUTION. Find a basis, such that two vectors v_1, v_2 are in the plane and such that a third vector v_3 is linearly independent to the first two. Since $(1,0,1), (0,1,1)$ are points in the plane and $(0,0,0)$ is in the plane, we can choose $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ which is perpendicular to the plane.

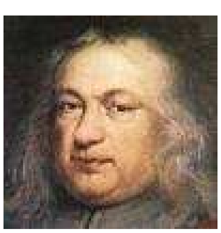
EXAMPLE. Find the coordinates of $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \{v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$. We have $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Therefore $[v]_{\mathcal{B}} = S^{-1}v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Indeed $-1v_1 + 3v_2 = v$.

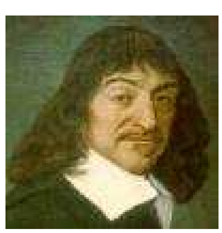
$\mathcal{B}$-MATRIX. If $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis in $\mathbf{R}^n$ and T is transformation from $R^n \to R^n$, then the $\mathcal{B}$-matrix of T is defined as

$$B = \begin{bmatrix} | & \dots & | \\ [T(v_1)]_{\mathcal{B}} & \dots & [T(v_n)]_{\mathcal{B}} \\ | & \dots & | \end{bmatrix}$$

The $\mathcal{B}$ matrix of A is $B = S^{-1}AS$.

COORDINATES HISTORY. Cartesian geometry was introduced by Fermat (1601-1665) and Descartes (1596-1650) around 1636. The introduction of algebraic methods into geometry had a huge influence on mathematics. The beginning of the vector concept came only later at the beginning of the 19'th Century with the work of Bolzano (1781-1848). The full power of coordinates come into play if we allow to chose our coordinate system adapted to the situation. Descartes biography shows how far dedication to the teaching of mathematics can go: *In 1649 Queen Christina of Sweden persuaded Descartes to go to Stockholm. However the Queen wanted to draw tangents at 5 AM. in the morning and Descartes broke the habit of his lifetime of getting up at 11 o'clock. After only a few months in the cold northern climate, walking to the palace at 5 o'clock every morning, he died of pneumonia.*


Fermat


Descartes


Christina


Bolzano

EXAMPLE. Let T be the reflection at the plane $x+2y+3z=0$. Find the transformation matrix B in the basis $v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. Because $T(v_1) = v_1 = [e_1]_{\mathcal{B}}, T(v_2) = v_2 = [e_2]_{\mathcal{B}}, T(v_3) = -v_3 = -[e_3]_{\mathcal{B}}$, the solution is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

SIMILARITY. If $B = S^{-1}AS$ then B is **similar** to A.

EXAMPLE. If A is similar to B, then A^2+A+1 is similar to $B^2+B+\mathbf{1}$. $B = S^{-1}AS, B^2 = S^{-1}B^2S, S^{-1}S = \mathbf{1}$, $S^{-1}(A^2+A+1)S = B^2+B+\mathbf{1}$.

PROPERTIES OF SIMILARITY. A, B similar and B, C similar, then A, C are similar. If A is similar to B, then B is similar to A.

TRUE OR FALSE? A shear A along the x-axes is similar to a reflection B at the x axes. Answer: False. We have $B^2 = 1$ but not $A^2 = 1$. If we had $B = S^{-1}AS$, then $B^2 = S^{-1}A^2S = S^{-1}S = 1$.

BIAS. You can survive by knowing the "BIAS" formula

$$\boxed{\boxed{B = S^{-1}AS}}.$$

The transformation in standard coordinates.

$$\begin{array}{ccc} v & \stackrel{S}{\leftarrow} & w = [v]_{\mathcal{B}} \\ A\downarrow & & \downarrow B \\ Av & \stackrel{S^{-1}}{\rightarrow} & Bw \end{array}$$

The transformation in $\mathcal{B}$-coordinates.

QUESTION. Can the matrix A, which belongs to a projection from $\mathbf{R}^3$ to a plane $x+y+6z=0$ be similar to a matrix which is a rotation by 20 degrees around the z axis? No: a non-invertible A can not be similar to an invertible B: if it were, the inverse $A = SBS^{-1}$ would exist: $A^{-1} = SB^{-1}S^{-1}$.

PROBLEM. Find a clever basis for the reflection of a light ray at the line $x+2y=0$. $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
SOLUTION. You can achieve $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

PROBLEM. Are all shears $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ with $a \neq 0$ similar? Yes, use a basis $v_1 = ae_1$ and $v_2 = e_2$.

SUMMARY: You can survive by knowing three things:

1) Build the matrix S by sticking the basis into the columns.
2) The coordinates of a vector v in the new basis are $S^{-1}v$.
3) A matrix A describing T in the standard basis becomes $B = S^{-1}AS$ in the new basis.

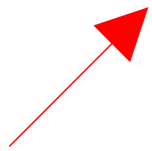
You can remember it as follows: you are the vector v, the basis are your new clothes. They hang in the closet S. If you change your cloths, you turn your closet inside out, then dress it. To do something in the new cloths first dress the old clothes, do it the old way, then get again into the new clothes.

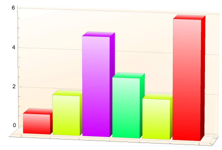
Very often, we need to do the reverse: in order to find a transformation A, we

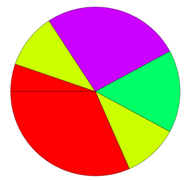
1) Find a basis suitable to the situation.
2) Find the matrix B in that situation (usually simple like diagonal)
3) Find $A = SBS^{-1}$.

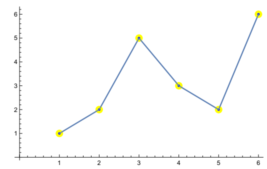
FROM VECTORS TO FUNCTIONS AND MATRICES. Vectors can be displayed in different ways:

$$\begin{bmatrix} 1 \\ 2 \\ 5 \\ 3 \\ 2 \\ 6 \end{bmatrix}$$









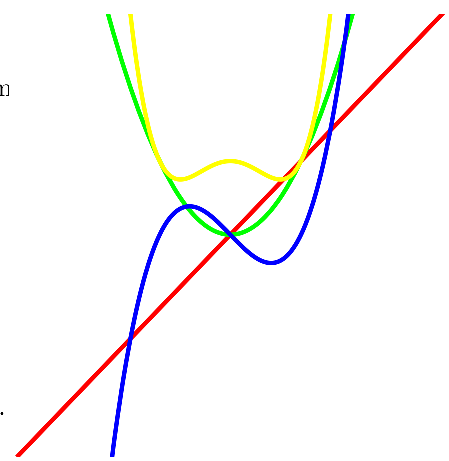
The values $(i, \vec{v}_i)$ of this bar chart can be interpreted as the graph of a **function** $f : 1,2,3,4,5,6 \to \mathbf{R}$, where $f(i) = \vec{v}_i$.
Also matrices can be treated as functions, but as a function of two variables. If M is a 8×8 matrix for example, we get a function $f(i,j) = [M]_{ij}$ which assigns to each square of the 8×8 checkerboard a number.

LINEAR SPACES. A space X of functions, vectors or matrices is called a **linear space** if (i) the 0 is in X, (ii) λv is in X if v is in X and (iii) $v + w$ is in X if v and w is in X.

There are three important types of linear spaces:
EUCLIDEAN SPACE: Subspaces X of R^n.
FUNCTION SPACE: The space of all functions in one variable.
SPACE OF MATRICES: X is the space of all $n \times m$ matrices.

EXAMPLES.

- The n-dimensional space R^n.
- linear subspaces of R^n like the trivial space $\{0\}$, the kernel or im
- M_n, the space of all square $n \times n$ matrices.
- P_n, the space of all polynomials of degree n.
- The space P of all polynomials.
- C^∞, the space of all smooth functions on the line
- C^0, the space of all continuous functions on the line.
- The space of all smooth vector fields in three dimensional space.
- C^1, the space of all differentiable functions on the line.
- L^2 the space of all functions for which $\int_\infty^\infty f^2(x)\, dx < \infty$.



ZERO VECTOR. The function f for which is everywhere equal to 0 is called the **zero function**. It plays the role of the zero vector in R^n. If we add this function to an other function g we get $0 + g = g$.

Careful, the **roots** of a function have nothing to do with the zero function. You should think of the roots of a function like as zero entries of a vector. For the zero vector, all entries have to be zero. For the zero function, all values $f(x)$ are zero.

CHECK: For subsets X of a function space, of for a subset of matrices R^n, we can check three properties to see whether the space is a linear space:

i) if x, y are in X, then $x + y$ is in X.
ii) If x is in X and λ is a real number, then λx is in X.
iii) 0 is in X.

WHICH OF THE FOLLOWING ARE LINEAR SPACES?

The space X of all polynomials of degree exactly 4.

The space X of all continuous functions on the unit interval $[-1, 1]$ which satisfy $f(0) = 1$.

The space X of all smooth functions satisfying $f(x + 1) = f(x)$. Example $f(x) = \sin(2\pi x) + \cos(6\pi x)$.

The space $X = \sin(x) + C^\infty(\mathbf{R})$ of all smooth functions $f(x) = \sin(x) + g$, where g is a smooth function.

The space X of all trigonometric polynomials $f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx)$.

The space X of all smooth functions on $\mathbf{R}$ which satisfy $f(1) = 1$. It contains for example $f(x) = 1 + \sin(x) + x$.

The space X of all continuous functions on $\mathbf{R}$ which satisfy $f(2) = 0$ and $f(10) = 0$.

The space X of all smooth functions on $\mathbf{R}$ which satisfy $\lim_{|x|\to\infty} f(x) = 0$.

The space X of all continuous functions on $\mathbf{R}$ which satisfy $\lim_{|x|\to\infty} f(x) = 1$.

The space X of all smooth functions on $\mathbf{R}^2$.

The space X of all 2×2 rotation dilation matrices

The space X of all upper triangular 3×3 matrices.

The space X of all 2×2 matrices A for which $A_{11} = 1$.

If you have seen multivariable calculus you can look at the following examples:

The space X of all vector fields (P, Q) in the plane, for which the curl $Q_x - P_y$ is zero everywhere.

The space X of all vector fields (P, Q, R) in space, for which the divergence $P_x + Q_y + R_z$ is zero everywhere.

The space X of all vector fields (P, Q) in the plane for which the line integral $\int_C F \cdot dr$ along the unit circle is zero.

The space X of all vector fields (P, Q, R) in space for which the flux through the unit sphere is zero.

The space X of all functions $f(x, y)$ of two variables for which $\int_0^1 \int_0^1 f(x, y)\, dxdy = 0$.

ORTHOGONAL PROJECTIONS	**Math 21b, O. Knill**

ORTHOGONALITY. Two vectors $\vec{v}$ and $\vec{w}$ are called **orthogonal** if $\vec{v} \cdot \vec{w} = 0$.
Examples. 1) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ are orthogonal in $\mathbf{R}^2$. 2) $\vec{v}$ and w are both orthogonal to the cross product $\vec{v} \times \vec{w}$ in $\mathbf{R}^3$.
$\vec{v}$ is called a **unit vector** if $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = 1$. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ are called **orthogonal** if they are pairwise orthogonal. They are called **orthonormal** if they are also unit vectors. A basis is called an **orthonormal basis** if it is a basis which is orthonormal. For an orthonormal basis, the matrix $A_{ij} = \vec{v}_i \cdot \vec{v}_j$ is the unit matrix.

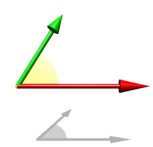
FACT. Orthogonal vectors are linearly independent and n orthogonal vectors in $\mathbf{R}^n$ form a basis.
Proof. The dot product of a **linear relation** $a_1\vec{v}_1 + \ldots + a_n\vec{v}_n = 0$ with $\vec{v}_k$ gives $a_k\vec{v}_k \cdot \vec{v}_k = a_k||\vec{v}_k||^2 = 0$ so that $a_k = 0$. If we have n linear independent vectors in $\mathbf{R}^n$, they automatically span the space.

ORTHOGONAL COMPLEMENT. A vector $\vec{w} \in \mathbf{R}^n$ is called **orthogonal** to a linear space V, if $\vec{w}$ is orthogonal to every vector $\vec{v} \in V$. The **orthogonal complement** of a linear space V is the set W of all vectors which are orthogonal to V. It forms a linear space because $\vec{v} \cdot \vec{w}_1 = 0, \vec{v} \cdot \vec{w}_2 = 0$ implies $\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = 0$.

ORTHOGONAL PROJECTION. The **orthogonal projection** onto a linear space V with **orthnormal** basis $\vec{v}_1, \dots, \vec{v}_n$ is the linear map $\boxed{P(\vec{x}) = \text{proj}_V(x) = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \cdots + (\vec{v}_n \cdot \vec{x})\vec{v}_n}$. (If you have not taken multi-variable calculus, take this as the definition and just check that $x - P(x)$ is perpendicular to x We can write this as $\boxed{P = QQ^T}$, where Q^T is the transpose matrix, obtained from Q by placing the rows of A as columns.

PYTHAGORAS: If $\vec{x}$ and $\vec{y}$ are orthogonal, then $||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$. Proof. Expand $(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y})$.
PROJECTIONS DO NOT INCREASE LENGTH: $||\text{proj}_\text{V}(\vec{x})|| \leq ||\vec{x}||$. Proof. Use Pythagoras: on $\vec{x} = \text{proj}_\text{V}(\vec{x}) + (\vec{x} - \text{proj}_\text{V}(\vec{x}))$. If $||\text{proj}_\text{V}(\vec{x})|| = ||\vec{x}||$, then $\vec{x}$ is in V.
CAUCHY-SCHWARTZ INEQUALITY: $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}|| \, ||\vec{y}||$. Proof: $\vec{x} \cdot \vec{y} = ||\vec{x}||||\vec{y}|| \cos(\alpha)$.
If $|\vec{x} \cdot \vec{y}| = ||\vec{x}||||\vec{y}||$, then $\vec{x}$ and $\vec{y}$ are parallel.
TRIANGLE INEQUALITY: $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$. Proof: $(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = ||\vec{x}||^2 + ||\vec{y}||^2 + 2\vec{x} \cdot \vec{y} \leq ||\vec{x}||^2 + ||\vec{y}||^2 + 2||\vec{x}||||\vec{y}|| = (||\vec{x}|| + ||\vec{y}||)^2$.

ANGLE. The **angle** between two vectors $\vec{x}, \vec{y}$ is $\alpha = \arccos\left(\frac{\vec{x} \cdot \vec{y}}{||\vec{x}||||\vec{y}||}\right)$.
CORRELATION. $\cos(\alpha) = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}||||\vec{y}||} \in [-1, 1]$ is the **correlation** of $\vec{x}$ and $\vec{y}$ if the vectors $\vec{x}, \vec{y}$ represent data of zero mean.



EXAMPLE. The angle between two orthogonal vectors is 90 degrees or 270 degrees. If $\vec{x}$ and $\vec{y}$ represent data of zero average then $\frac{\vec{x} \cdot \vec{y}}{||\vec{x}||||\vec{y}||}$ is called the **statistical correlation** of the data.

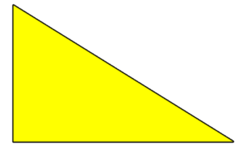
QUESTION. Express the fact that $\vec{x}$ is in the kernel of a matrix A using orthogonality.
ANSWER: $A\vec{x} = 0$ means that $\vec{w}_k \cdot \vec{x} = 0$ for every row vector $\vec{w}_k$ of $\mathbf{R}^n$.
REMARK. We will call later the matrix A^T, obtained by switching rows and columns of A the **transpose** of A. You see already that the image of A^T is orthogonal to the kernel of A.

QUESTION. Find a basis for the orthogonal complement of the linear space V spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$.

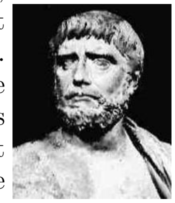
ANSWER: The orthogonality of $\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$ to the two vectors means solving the linear system of equations $x + 2y + 3z + 4w = 0, 4x + 5y + 6z + 7w = 0$. An other way to solve it: the kernel of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ is the orthogonal complement of V. This reduces the problem to an older problem.

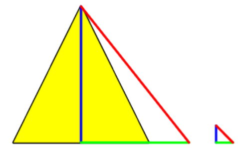
ON THE RELEVANCE OF ORTHOGONALITY.

1) From -2800 til -2300 BC, Egyptians used ropes divided into length ratios like 3 : 4 : 5 to build triangles. This allowed them to triangulate areas quite precisely: for example to build irrigation needed because the Nile was reshaping the land constantly or to build the pyramids: for the **great pyramid at Giza** with a base length of 230 meters, the average error on each side is less then 20cm, an error of less then 1/1000. A key to achieve this was **orthogonality**.

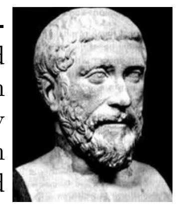


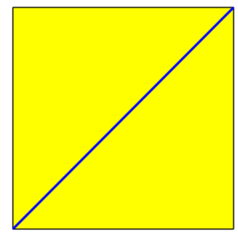
2) During one of Thales (-624 BC to (-548 BC)) journeys to Egypt, he used a geometrical trick to **measure the height** of the great pyramid. He measured the size of the shadow of the pyramid. Using a stick, he found the relation between the length of the stick and the length of its shadow. The same length ratio applies to the pyramid (**orthogonal** triangles). Thales found also that triangles inscribed into a circle and having as the base as the diameter must have a right angle.



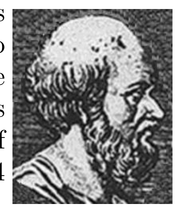


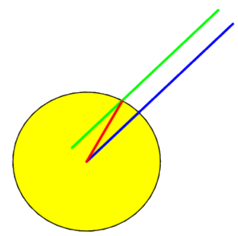
3) The Pythagoreans (-572 until -507) were interested in the discovery that the squares of a lengths of a triangle with two **orthogonal** sides would add up as $a^2 + b^2 = c^2$. They were puzzled in assigning a length to the diagonal of the unit square, which is $\sqrt{2}$. This number is irrational because $\sqrt{2} = p/q$ would imply that $q^2 = 2p^2$. While the prime factorization of q^2 contains an even power of 2, the prime factorization of $2p^2$ contains an odd power of 2.



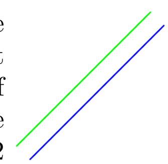


4) Eratosthenes (-274 until 194) realized that while the sun rays were **orthogonal** to the ground in the town of Scene, this did no more do so at the town of Alexandria, where they would hit the ground at 7.2 degrees). Because the distance was about 500 miles and 7.2 is 1/50 of 360 degrees, he measured the circumference of the earth as 25'000 miles - pretty close to the actual value 24'874 miles.

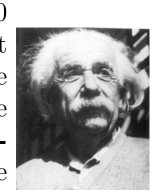




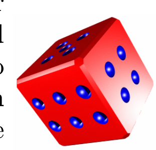
5) Closely related to **orthogonality** is **parallelism**. Mathematicians tried for ages to prove Euclid's parallel axiom using other postulates of Euclid (-325 until -265). These attempts had to fail because there are geometries in which parallel lines always meet (like on the sphere) or geometries, where parallel lines never meet (the Poincaré half plane). Also these geometries can be studied using linear algebra. The geometry on the sphere with **rotations**, the geometry on the half plane uses Möbius transformations, 2×2 matrices with determinant one.



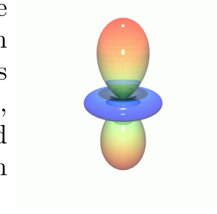
6) The question whether the angles of a right triangle do in reality always add up to 180 degrees became an issue when geometries where discovered, in which the measurement depends on the position in space. Riemannian geometry, founded 150 years ago, is the foundation of **general relativity**, a theory which describes gravity geometrically: the presence of mass bends space-time, where the dot product can depend on space. **Orthogonality** becomes relative too. On a sphere for example, the three angles of a triangle are bigger than 180^+. Space is curved.



7) In **probability theory** the notion of **independence** or **decorrelation** is used. For example, when throwing a dice, the number shown by the first dice is independent and decorrelated from the number shown by the second dice. Decorrelation is identical to **orthogonality**, when vectors are associated to the random variables. The **correlation coefficient** between two vectors $\vec{v}, \vec{w}$ is defined as $\vec{v} \cdot \vec{w}/(|\vec{v}||\vec{w}|)$. It is the cosine of the angle between these vectors.

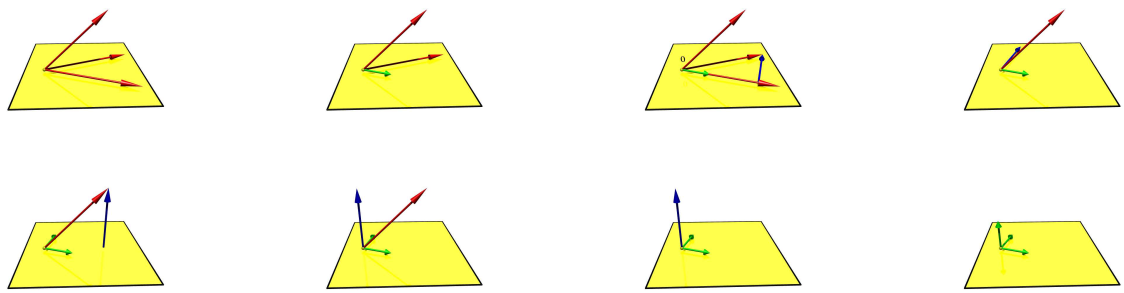


8) In **quantum mechanics**, states of atoms are described by functions in a linear space of functions. The states with energy $-E_B/n^2$ (where $E_B = 13.6eV$ is the Bohr energy) in a hydrogen atom. States in an atom are **orthogonal**. Two states of two different atoms which don't interact are **orthogonal**. One of the challenges in quantum computation, where the computation deals with qubits (=vectors) is that orthogonality is not preserved during the computation (because we don't know all the information). Different states can interact.



GRAM SCHMIDT AND QR FACTORIZATION — Math 21b, O. Knill

GRAM-SCHMIDT PROCESS.
Let $\vec{v}_1, \dots, \vec{v}_n$ be a basis in V. Let $\vec{w}_1 = \vec{v}_1$ and $\vec{u}_1 = \vec{w}_1/||\vec{w}_1||$. The Gram-Schmidt process recursively constructs from the already constructed orthonormal set $\vec{u}_1, \dots, \vec{u}_{i-1}$ which spans a linear space V_{i-1} the new vector $\vec{w}_i = (\vec{v}_i - \mathrm{proj}_{V_{i-1}}(\vec{v}_i))$ which is orthogonal to V_{i-1}, and then normalizes $\vec{w}_i$ to get $\vec{u}_i = \vec{w}_i/||\vec{w}_i||$. Each vector $\vec{w}_i$ is orthonormal to the linear space V_{i-1}. The vectors $\{\vec{u}_1, \dots, \vec{u}_n\}$ form then an orthonormal basis in V.



EXAMPLE.
Find an orthonormal basis for $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

SOLUTION.

1. $\vec{u}_1 = \vec{v}_1/||\vec{v}_1|| = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

2. $\vec{w}_2 = (\vec{v}_2 - \mathrm{proj}_{V_1}(\vec{v}_2)) = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$. $\vec{u}_2 = \vec{w}_2/||\vec{w}_2|| = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

3. $\vec{w}_3 = (\vec{v}_3 - \mathrm{proj}_{V_2}(\vec{v}_3)) = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$, $\vec{u}_3 = \vec{w}_3/||\vec{w}_3|| = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

QR FACTORIZATION.
The formulas can be written as

$$\vec{v}_1 = ||\vec{v}_1||\vec{u}_1 = r_{11}\vec{u}_1$$

$$\cdots$$

$$\vec{v}_i = (\vec{u}_1 \cdot \vec{v}_i)\vec{u}_1 + \cdots + (\vec{u}_{i-1} \cdot \vec{v}_i)\vec{u}_{i-1} + ||\vec{w}_i||\vec{u}_i = r_{1i}\vec{u}_1 + \cdots + r_{ii}\vec{u}_i$$

$$\cdots$$

$$\vec{v}_n = (\vec{u}_1 \cdot \vec{v}_n)\vec{u}_1 + \cdots + (\vec{u}_{n-1} \cdot \vec{v}_n)\vec{u}_{n-1} + ||\vec{w}_n||\vec{u}_n = r_{1n}\vec{u}_1 + \cdots + r_{nn}\vec{u}_n$$

which means in matrix form

$$A = \begin{bmatrix} | & | & \cdot & | \\ \vec{v}_1 & \cdots & \cdot & \vec{v}_n \\ | & | & \cdot & | \end{bmatrix} = \begin{bmatrix} | & | & \cdot & | \\ \vec{u}_1 & \cdots & \cdot & \vec{u}_n \\ | & | & \cdot & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdot & r_{1n} \\ 0 & r_{22} & \cdot & r_{2n} \\ 0 & 0 & \cdot & r_{nn} \end{bmatrix} = QR\,,$$

where A and Q are $m \times n$ matrices and R is a $n \times n$ matrix which has $r_{ij} = \vec{v}_j \cdot \vec{u}_i$, for $i < j$ and $v_{ii} = |\vec{v}_i|$

THE GRAM-SCHMIDT PROCESS PROVES: Any matrix A with linearly independent columns $\vec{v}_i$ can be decomposed as $A = QR$, where Q has orthonormal column vectors and where R is an upper triangular square matrix with the same number of columns than A. The matrix Q has the orthonormal vectors $\vec{u}_i$ in the columns.

In the example, the matrix with the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$.

$\vec{v}_1 = ||\vec{v}_1||\vec{u}_1$
$\vec{v}_2 = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 + ||\vec{w}_2||\vec{u}_2$
$\vec{v}_3 = (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 + (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2 + ||\vec{w}_3||\vec{u}_3,$

so that $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$.

PRO MEMORIA.
While building the matrix R we keep track of the vectors w_i during the Gram-Schmidt procedure. At the end you have vectors $\vec{u}_i$ and the matrix R has $\boxed{||\vec{w}_i||}$ in the diagonal as well as the dot products $\boxed{\vec{u}_i \cdot \vec{v}_j}$ in the upper right triangle where $i < j$.

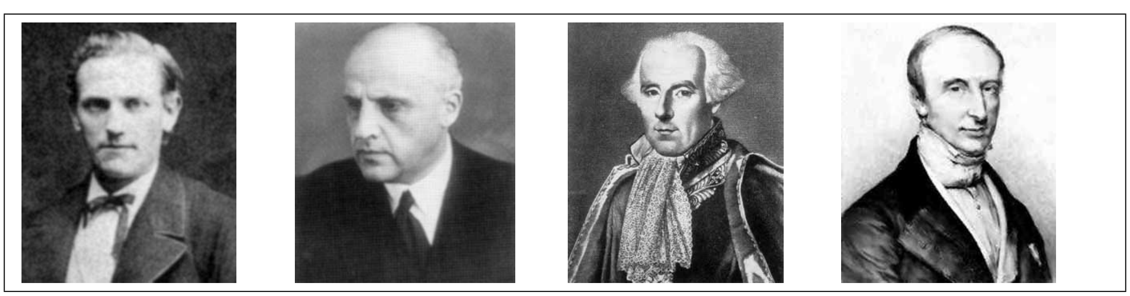
PROBLEM. Make the QR decomposition of $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. $\vec{w}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. $\vec{w}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.
$\vec{u}_2 = \vec{w}_2$. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = QR$.

WHY do we care to have an orthonormal basis?

- An orthonormal basis looks like the standard basis $\vec{v}_1 = (1, 0, \dots, 0), \dots, \vec{v}_n = (0, 0, \dots, 1)$. Actually, we will see that an orthonormal basis into a standard basis or a mirror of the standard basis.
- The Gram-Schmidt process is tied to the factorization $A = QR$. The later helps to solve linear equations. In physical problems like in astrophysics, the numerical methods to simulate the problems one needs to invert huge matrices in every time step of the evolution. The reason why this is necessary sometimes is to assure the numerical method is stable implicit methods. Inverting $A^{-1} = R^{-1}Q^{-1}$ is easy because R and Q are easy to invert.
- For many physical problems like in quantum mechanics or mechanical systems, the matrices which occur are **symmetric** $A^* = A$, where $A^*_{ij} = \overline{A}_{ji}$. For such matrices, there will a natural orthonormal basis.
- The **formula for the projection** onto a linear subspace V simplifies with an orthonormal basis $\vec{v}_j$ in V: $\mathrm{proj}_V(\vec{x}) = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \cdots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$.
- An orthonormal basis simplifies computations due to the presence of many zeros $\vec{w}_j \cdot \vec{w}_i = 0$. This is especially the case for problems with symmetry.
- The Gram Schmidt process allows to get classes of classical polynomials, which are important in physics: Chebyshev polynomials, Laguerre polynomials or Hermite polynomials.
- QR-factorization allows fast computation of the determinant, least square solutions $R^{-1}Q^{-1}\vec{b}$ of overdetermined systems $A\vec{x} = \vec{b}$ or finding eigenvalues - all topics which will appear later.

SOME HISTORY.

The recursive formulae of the process were stated by Erhard Schmidt (1876-1959) in 1907. The essence of the formulae were already in a 1883 paper of J.P.Gram in 1883 which Schmidt mentions in a footnote. The process seems already have been used by Laplace (1749-1827) and was also used by Cauchy (1789-1857) in 1836.



Gram Schmidt Laplace Cauchy

ORTHOGONAL MATRICES — Math 21b, O. Knill

TRANSPOSE The **transpose** of a matrix A is the matrix $(A^T)_{ij} = A_{ji}$. If A is a $n \times m$ matrix, then A^T is a $m \times n$ matrix. For square matrices, the transposed matrix is obtained by reflecting the matrix at the diagonal. In general, the rows of A^T are the columns of A.

EXAMPLES The transpose of a vector $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the row vector $A^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

PROPERTIES (we write $v \cdot w = v^T.w$ for the dot product.)

a) $(AB)^T = B^T A^T$.
b) $v^T w$ is the dot product $\vec{v} \cdot \vec{w}$.
c) $\vec{x} \cdot A\vec{y} = A^T\vec{x} \cdot \vec{y}$.
d) $(A^T)^T = A$.
e) $(A^T)^{-1} = (A^{-1})^T$ for invertible A

PROOFS.
a) $(AB)^T_{kl} = (AB)_{lk} = \sum_i A_{li}B_{ik} = \sum_i B^T_{ki}A^T_{il} = (B^T A^T)_{kl}$.
b) by definition.
c) $x \cdot Ay = x^T Ay = (A^T x)^T y = A^T x \cdot y$.
d) $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$.
e) $1_n = 1_n^T = (AA^{-1})^T = (A^{-1})^T A^T$ using a).

ORTHOGONAL MATRIX. A $n \times n$ matrix A is called **orthogonal** if $A^T A = 1_n$. The corresponding linear transformation is called **orthogonal**.

INVERSE. It is easy to invert an orthogonal matrix because $A^{-1} = A^T$.

EXAMPLES. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ is orthogonal because its column vectors have length 1 and are orthogonal to each other. Indeed: $A^T A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A reflection at a line is an orthogonal transformation because the columns of the matrix A have length 1 and are orthogonal. Indeed: $A^T A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

PRESERVATION OF LENGTH AND ANGLE. Orthogonal transformations preserve the dot product: $\boxed{A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}}$ Proof. $A\vec{x} \cdot A\vec{y} = A^T A\vec{x} \cdot \vec{y}$ and because of the orthogonality property, this is $\vec{x} \cdot \vec{y}$.

> Orthogonal transformations preserve the **length** of vectors as well as the **angles** between them.

Proof. We have $||A\vec{x}||^2 = A\vec{x} \cdot A\vec{x} = \vec{x} \cdot \vec{x}\ ||\vec{x}||^2$. Let α be the angle between $\vec{x}$ and $\vec{y}$ and let β denote the angle between $A\vec{x}$ and $A\vec{y}$ and α the angle between $\vec{x}$ and $\vec{y}$. Using $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ we get $||A\vec{x}||||A\vec{y}|| \cos(\beta) = A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y} = ||\vec{x}||||\vec{y}|| \cos(\alpha)$. Because $||A\vec{x}|| = ||\vec{x}||, ||A\vec{y}|| = ||\vec{y}||$, this means $\cos(\alpha) = \cos(\beta)$. Because this property holds for all vectors we can rotate $\vec{x}$ in plane V spanned by $\vec{x}$ and $\vec{y}$ by an angle ϕ to get $\cos(\alpha + \phi) = \cos(\beta + \phi)$ for all ϕ. Differentiation with respect to ϕ at $\phi = 0$ shows also $\sin(\alpha) = \sin(\beta)$ so that $\alpha = \beta$.

ORTHOGONAL MATRICES AND BASIS. A linear transformation A is orthogonal if and only if the column vectors of A form an orthonormal basis.
Proof. Look at $A^T A = I_n$. Each entry is a dot product of a column of A with an other column of A.

COMPOSITION OF ORTHOGONAL TRANSFORMATIONS. The composition of two orthogonal transformations is orthogonal. The inverse of an orthogonal transformation is orthogonal. Proof. The properties of the transpose give $(AB)^T AB = B^T A^T AB = B^T B = 1$ and $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = 1_n$.

EXAMPLES.
The composition of two reflections at a line is a rotation.
The composition of two rotations is a rotation.
The composition of a reflections at a plane with a reflection at an other plane is a rotation (the axis of rotation is the intersection of the planes).

ORTHOGONAL PROJECTIONS. The orthogonal projection P onto a linear space with orthonormal basis $\vec{v}_1, \dots, \vec{v}_n$ is the matrix $\boxed{AA^T}$, where A is the matrix with column vectors $\vec{v}_i$. To see this just translate the formula $P\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$ into the language of matrices: $A^T\vec{x}$ is a vector with components $\vec{b}_i = (\vec{v}_i \cdot \vec{x})$ and $A\vec{b}$ is the sum of the $\vec{b}_i\vec{v}_i$, where $\vec{v}_i$ are the column vectors of A.
Orthogonal the only projections which is orthogonal is the identity!

EXAMPLE. Find the orthogonal projection P from $\mathbf{R}^3$ to the linear space spanned by $\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \frac{1}{5}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Solution: $AA^T = \begin{bmatrix} 0 & 1 \\ 3/5 & 0 \\ 4/5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9/25 & 12/25 \\ 0 & 12/25 & 16/25 \end{bmatrix}$.

WHY ARE ORTHOGONAL TRANSFORMATIONS USEFUL?

- In Physics, Galileo transformations are compositions of translations with orthogonal transformations. The laws of classical mechanics are invariant under such transformations. This is a symmetry.
- Many coordinate transformations are orthogonal transformations. We will see examples when dealing with differential equations.
- In the QR decomposition of a matrix A, the matrix Q is orthogonal. Because $Q^{-1} = Q^t$, this allows to invert A easier.
- Fourier transformations are orthogonal transformations. We will see this transformation later in the course. In application, it is useful in computer graphics (like JPG) and sound compression (like MP3).

WHICH OF THE FOLLOWING MAPS ARE ORTHOGONAL TRANSFORMATIONS?:

Yes	No	Shear in the plane.
Yes	No	Projection in three dimensions onto a plane.
Yes	No	Reflection in two dimensions at the origin.
Yes	No	Reflection in three dimensions at a plane.
Yes	No	Dilation with factor 2.
Yes	No	The Lorenz boost $\vec{x} \mapsto A\vec{x}$ in the plane with $A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$
Yes	No	A translation.

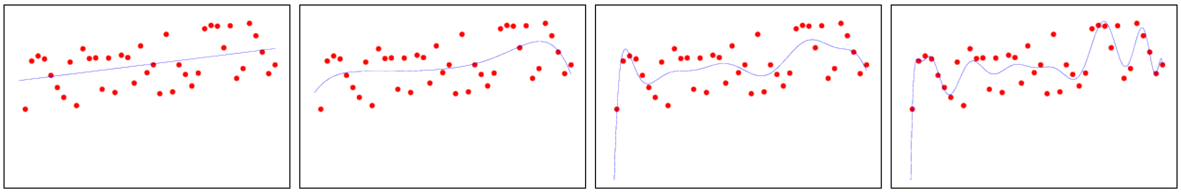
CHANGING COORDINATES ON THE EARTH. Problem: what is the matrix which rotates a point on earth with (latitude,longitude)=(a_1, b_1) to a point with (latitude,longitude)=(a_2, b_2)? Solution: The matrix which rotate the point $(0,0)$ to (a,b) a composition of two rotations. The first rotation brings the point into the right latitude, the second brings the point into the right longitude. $R_{a,b} = \begin{bmatrix} \cos(b) & -\sin(b) & 0 \\ \sin(b) & \cos(b) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(a) & 0 & -\sin(a) \\ 0 & 1 & 0 \\ \sin(a) & 0 & \cos(a) \end{bmatrix}$. To bring a point (a_1, b_1) to a point (a_2, b_2), we form $A = R_{a_2,b_2} R^{-1}_{a_1,b_1}$.

EXAMPLE: With Cambridge (USA): $(a_1, b_1) = (42.366944, 288.893889)\pi/180$ and Zürich (Switzerland): $(a_2, b_2) = (47.377778, 8.551111)\pi/180$, we get the matrix

$$A = \begin{bmatrix} 0.178313 & -0.980176 & -0.0863732 \\ 0.983567 & 0.180074 & -0.0129873 \\ 0.028284 & -0.082638 & 0.996178 \end{bmatrix}.$$

LEAST SQUARES AND DATA — Math 21b, O. Knill

The best possible "solution" of an inconsistent linear systems $Ax = b$ is called the **least square solution**. It is the orthogonal projection of b onto the image $\mathrm{im}(A)$ of A. What we know about the kernel and the image of linear transformations helps to understand this and will leads to an explicit formulas for the least square fit. Why do we care about non-consistent systems? Often we have to solve linear systems of equations with more constraints than variables. An example is when we try to find the best polynomial which passes through a set of points. This problem is called **data fitting**. If we wanted to accommodate all data, the degree of the polynomial would become too large. The fit would look too wiggly. Taking a smaller degree polynomial will not only be more convenient but also give a better picture. Especially important is **regression**, the fitting of data with lines.



The above pictures show 30 data points which are fitted best with polynomials of degree 1, 6, 11 and 16. The first linear fit maybe tells most about the trend of the data.

The orthogonal complement The image $\mathrm{im}(A)$ is a linear space. Because a vector is in the kernel of A^T if and only if it is orthogonal to the rows of A^T and so to the columns of A, the kernel of A^T is the orthogonal complement of $\mathrm{im}(A)$: $\boxed{(\mathrm{im}(A))^\perp = \ker(A^T)}$

Examples.

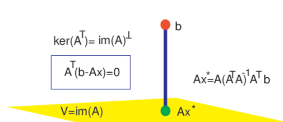
1) $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. The kernel V of $A^T = \begin{bmatrix} a & b & c \end{bmatrix}$ consists of all vectors satisfying $ax + by + cz = 0$. V is a plane. The orthogonal complement is the image of A which is spanned by the normal vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to the plane.

2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The image of A is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ the kernel of A^T is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Orthogonal projection. If $\vec{b}$ is a vector and V is a linear subspace, then $\mathrm{proj}_V(\vec{b})$ is the vector closest to $\vec{b}$ on V: given any other vector $\vec{v}$ on V, one can form the triangle $\vec{b}, \vec{v}, \mathrm{proj}_V(\vec{b})$ which has a right angle at $\mathrm{proj}_V(\vec{b})$ and invoke Pythagoras.

Lemma: For any $m \times n$ matrix $\boxed{\ker(A) = \ker(A^T A)}$ Proof. $\subset$ is clear. On the other hand $A^T A v = 0$ means that Av is in the kernel of A^T. But since the image of A is orthogonal to the kernel of A^T, we have $A\vec{v} = 0$, which means $\vec{v}$ is in the kernel of A.

Least square solution. The **least square solution** of $A\vec{x} = \vec{b}$ is the vector $\vec{x}^*$ such that $A\vec{x}^*$ is closest to $\vec{b}$ from all other vectors $A\vec{x}$. In other words, $A\vec{x}^* = \mathrm{proj}_V(\vec{b})$, where $V = \mathrm{im}(V)$. Because $\vec{b} - A\vec{x}^*$ is in $V^\perp = \mathrm{im}(A)^\perp = \ker(A^T)$, we have $A^T(\vec{b} - A\vec{x}^*) = 0$. The last equation means that $\vec{x}^*$ is a solution of $\boxed{A^T A\vec{x} = A^T\vec{b}, \text{ the } \textbf{normal equation} \text{ of } A\vec{x} = \vec{b}}$. If the kernel of A is trivial, then the kernel of $A^T A$ is trivial and $A^T A$ can be inverted. Therefore $\boxed{\vec{x}^* = (A^T A)^{-1} A^T \vec{b} \text{ is the least square solution.}}$



Why do we like least square solutions? If $\vec{x}^*$ is the least square solution of $A\vec{x} = \vec{b}$ then $||A\vec{x}^* - \vec{b}|| \leq ||A\vec{x} - \vec{b}||$ for all $\vec{x}$.
Proof. $A^T(A\vec{x}^* - \vec{b}) = 0$ means that $A\vec{x}^* - \vec{b}$ is in the kernel of A^T which is orthogonal to $V = \mathrm{im}(A)$. That is $\mathrm{proj}_V(\vec{b}) = A\vec{x}^*$ which is the closest point to $\vec{b}$ on V.

As an application we get a universal formula for the **orthogonal projection** If $\vec{v}_1, \dots, \vec{v}_n$ is a basis in V which is not necessarily orthonormal, then the orthogonal projection is $\boxed{\vec{x} \mapsto A(A^TA)^{-1}A^T(\vec{x})}$ where $A = [\vec{v}_1, \dots, \vec{v}_n]$.
Proof. $\vec{x} = (A^TA)^{-1}A^T\vec{b}$ is the least square solution of $A\vec{x} = \vec{b}$. Therefore $A\vec{x} = A(A^TA)^{-1}A^T\vec{b}$ is the vector in $\mathrm{im}(A)$ closest to $\vec{b}$.

Special case: If $\vec{w}_1, \dots, \vec{w}_n$ is an orthonormal basis in V, we had seen earlier that AA^T with $A = [\vec{w}_1, \dots, \vec{w}_n]$ is the orthogonal projection onto V (this was just rewriting $A\vec{x} = (\vec{w}_1 \cdot \vec{x})\vec{w}_1 + \cdots + (\vec{w}_n \cdot \vec{x})\vec{w}_n$ in matrix form.) This follows from the above formula because $A^TA = I$ in that case.

Example: let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$. The orthogonal projection onto $V = \mathrm{im}(A)$ is $\vec{b} \mapsto A(A^TA)^{-1}A^T\vec{b}$. We have $A^TA = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ and $A(A^TA)^{-1}A^T = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
For example, the projection of $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is $\vec{x}^* = \begin{bmatrix} 2/5 \\ 4/5 \\ 0 \end{bmatrix}$ and the distance to $\vec{b}$ is $1/\sqrt{5}$. The point $\vec{x}^*$ is the point on V which is closest to $\vec{b}$.

Remember the formula for the distance of $\vec{b}$ to a plane V with normal vector $\vec{n}$? It was $d = |\vec{n} \cdot \vec{b}|/||\vec{n}||$. In our case, we can take $\vec{n} = [-2, 1, 0]$ and get the distance $1/\sqrt{5}$. Let's check: the distance of $\vec{x}^*$ and $\vec{b}$ is $||(2/5, -1/5, 0)|| = 1/\sqrt{5}$.

example: Let $A = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. Problem: find the matrix of the orthogonal projection onto the image of A. The image of A is a one-dimensional line spanned by the vector $\vec{v} = (1, 2, 0, 1)$. We calculate $A^TA = 6$. Then

$$A(A^TA)^{-1}A^T = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}/6 = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}/6$$

DATA FIT. Find a quadratic polynomial $p(t) = at^2 + bt + c$ which best fits the four data points $(-1, 8), (0, 8), (1, 4), (2, 16)$.

$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix}^T$. $A^TA = \begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix}$ and $\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$.

Software packages like Mathematica have already built in the facility to fit numerical data:

```
DataPoints = {{-1,8},{0,8},{1,4},{2,16}}
f=Function[y,Fit[DataPoints,{1,x,x^2},x] /. x->y];
S=Show[{ListPlot[DataPoints], Plot[f[t],{t,-1,2}]}];
```

Mathematica or statistics packages do the same to find the fit then what we do: **"Solving" an inconsistent system of linear equations as best as possible.**

Here is an other theoretical **problem:** prove that $\mathrm{im}(A) = \mathrm{im}(AA^T)$.

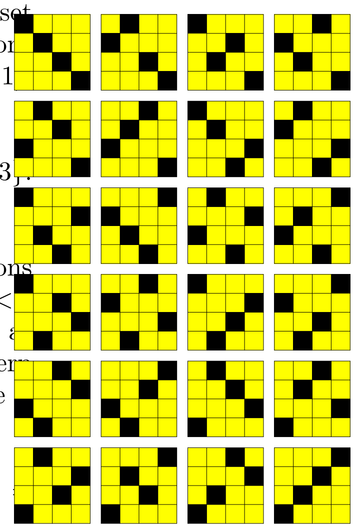
SOLUTION. The image of AA^T is contained in the image of A because we can write $\vec{v} = AA^T\vec{x}$ as $\vec{v} = A\vec{y}$ with $\vec{y} = A^T\vec{x}$. On the other hand, if $\vec{v}$ is in the image of A, then $\vec{v} = A\vec{x}$. If $\vec{x} = \vec{y} + \vec{z}$, where $\vec{y}$ in the kernel of A and $\vec{z}$ orthogonal to the kernel of A, then $A\vec{x} = A\vec{z}$. Because $\vec{z}$ is orthogonal to the kernel of A, it is in the image of A^T. Therefore, $\vec{z} = A^T\vec{u}$ and $\vec{v} = A\vec{z} = AA^T\vec{u}$ is in the image of AA^T.

A **permutation** of $\{1, 2, \dots, n\ \}$ is a rearrangement of the se $\{1, 2, \dots, n\}$. There are $n! = n \cdot (n-1) \cdots 1$ different permutatio of $\{1, 2, \dots, n\}$: fixing the position of first element leaves $(n-1$ possibilities to permute the rest.

There are 6 permutations of $\{1, 2, 3$: $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$.

The $n \times n$ matrix A which has zeros everywhere except at the positions $A_{\pi(i)i} = 1$ defines the **pattern** of π. An **up-crossing** is a pair $k <$ such that $\pi(k) < \pi(l)$. The **sign** of a permutation π is defined a $\text{sign}(\pi) = (-1)^u$, where u is the number of up-crossings in the patter of π. It is the number pairs of black squares, where the upper square to the right.

Examples: $\text{sign}(1, 2) = 0$, $\text{sign}(2, 1) = 1$. $\text{sign}(1, 2, 3) = \text{sign}(3, 2, 1)$ $\text{sign}(2, 3, 1) = 1$. $\text{sign}(1, 3, 2) = \text{sign}(3, 2, 1) = \text{sign}(2, 1, 3) = -1$.



The **determinant** of a $n \times n$ matrix $A = a_{ij}$ is defined as the sum

$$\boxed{\sum_\pi \text{sign}(\pi) a_{\pi(1)1} a_{\pi(2)2} \cdots a_{\pi(n)n}}$$

where π is a permutation of $\{1, 2, \dots, n\}$.

Example: the determinant of $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$. There are two permutations of $(1, 2)$. The identity permutation $(1, 2)$ gives $a_{11} a_{12}$, the permutation $(2, 1)$ gives $a_{21} a_{22}$.
If you have seen some multi-variable calculus, you know that $\det(A)$ is the area of the parallelogram spanned by the column vectors of A. The two vectors form a basis if and only if $\det(A) \neq 0$.

The determinant of $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is $aei + bfg + cdh - ceg - fha - bdi$. The 6 summands belong to the 6 permutations of $(1, 2, 3)$. Geometrically, $\det(A)$ is the volume of the parallelepiped spanned by the column vectors of A. We know from multivariable calculus that the three vectors form a basis if and only if $\det(A) \neq 0$.

By definition, the determinant of a pattern matrix is equal to the sign of the permutation.

To compute the determinant of a $n \times n$ matrices $A = a_{ij}$. Choose a column i. For each entry a_{ji} in that column, take the $(n-1) \times (n-1)$ matrix a_{ij} which does not contain the i'th column and j'th row. The determinant of A_{ij} is called a **minor**. One gets

$$\boxed{\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})}$$

This **Laplace expansion** is just a convenient arrangement of the permutations: listing all permutations of the form $(1, *, \dots, *)$ of n elements is the same then listing all permutations of $(2, *, \dots, *)$ of $(n-1)$ elements etc.

The determinant of a **diagonal** or **triangular** matrix is the product of its diagonal elements.

Example: $\det(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}) = 20$.

The determinant of a **partitioned matrix** $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the product $\det(A)\det(B)$. The reason is that we can can get a permutation by choosing a Example $\det(\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}) = 2 \cdot 12 = 24$.

If the columns of A and B are the same except for the i'th column,

$$\boxed{\det([v_1,...,v,...v_n]) + \det([v_1,...,w,...v_n]) = \det([v_1,...,v+w,...v_n])}$$

In general, one has $\boxed{\det([v_1,...,kv,...v_n]) = k \det([v_1,...,v,...v_n])}$ The same identities hold for rows and follow directly from the original definition of the determinant.

Determining rref(A) also determines $\det(A)$.
If A is a matrix and $\lambda_1,...,\lambda_k$ are the factors which are used to scale different rows and s is the total number of times, two rows were switched, then $\boxed{\det(A) = (-1)^s \alpha_1 \cdots \lambda_k \det(\text{rref}(A))}$.

Because of the last formula: $\boxed{\text{A } n \times n \text{ matrix } A \text{ is invertible if and only if } \det(A) \neq 0.}$

Find the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \end{bmatrix}$. Three row transpositions give $B = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ a matrix which has determinant 84. Therefore $\det(A) = (-1)^3 \det(B) = -84$.

Some properties will only be seen in the next lecture: $\boxed{\det(AB) = \det(A)\det(B)}$ $\boxed{\det(SAS^{-1}) = \det(A)}$ $\boxed{\det(A^{-1}) = \det(A)^{-1}}$ $\boxed{\det(A^T) = \det(A)}$

If B is obtained from A by switching two rows, then $\det(B) = -\det(A)$. If B is obtained by adding an other row to a given row, then this does not change the value of the determinant.

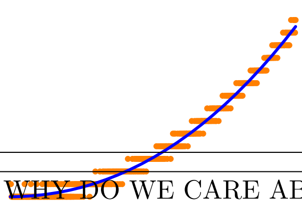
We will prove that $\det(AB) = \det(A)\det(B)$ next time: one brings the $n \times n$ matrix $[A|AB]$ into row reduced echelon form. Similar than the augmented matrix $[A|b]$ was brought into the form $[1|A^{-1}b]$, we end up with $[1|A^{-1}AB] = [1|B]$. By looking at the $n \times n$ matrix to the left during Gauss-Jordan elimination, the determinant has changed by a factor $\det(A)$. We end up with a matrix B which has determinant $\det(B)$. Therefore, $\det(AB) = \det(A)\det(B)$.
Proof of $\det(A^T) = \det(A)$: the transpose of a pattern is a pattern with the same signature.

Problem: determine $\det(A^{100})$, where A is the matrix $\begin{vmatrix} 1 & 2 \\ 3 & 16 \end{vmatrix}$.
Solution: $\det(A) = 10$, $\det(A^{100}) = (\det(A))^{100} = 10^{100} = 1 \cdot$ gogool. This name as well as the gogoolplex $= 10^{10^{100}}$ are official. They are huge numbers: the mass of the universe for example is $10^{52} kg$ and $1/10^{10^{51}}$ is the chance to find yourself on Mars by quantum fluctuations. (R.E. Crandall, Scientific American, Feb. 1997).

Example: Because $Q^T Q = 1$, we have $\det(Q)^2 = 1$ and so $|\det(Q)| = 1$ Rotations have determinant 1, reflections can have determinant -1.

Example: If $A = QR$, then $\det(A) = \det(Q)\det(R)$. The determinant of Q is ± 1, the determinant of R is the product of the diagonal elements of R.

How fast can we compute $\det(A)$?



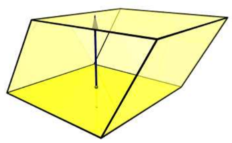
The cost to find the determinant is the same as for the Gauss-Jordan elimination. A measurements of the time needed for Mathematica to compute a determinant of a random $n \times n$ matrix. The matrix size ranged from n=1 to n=300. We see a cubic fit of these data.

WHY DO WE CARE ABOUT DETERMINANTS?

- determines invertibility of matrices
- geometric interpretation as volume
- explicit algebraic inversion of a matrix
- is a natural functional in particle and statistical physics
- define orientation in any dimensions
- change of variable formulas in higher dimensions
- alternative concepts are unnatural
- helps to figure out similarity of matrices

DETERMINANTS II — Math 21b, O.Knill

DETERMINANT AND VOLUME. If A is a $n \times n$ matrix, then $|\det(A)|$ is the volume of the n-dimensional parallelepiped E_n spanned by the n column vectors v_j of A.

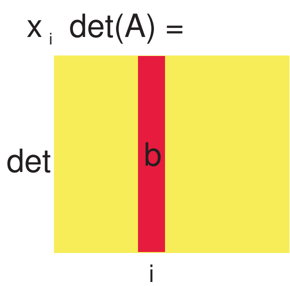


Proof. Use the QR decomposition $A = QR$, where Q is orthogonal and R is upper triangular. From $QQ^T = 1_n$, we get $1 = \det(Q)\det(Q^T) = \det(Q)^2$ see that $|\det(Q)| = 1$. Therefore, $\det(A) = \pm\det(R)$. The determinant of R is the product of the $||u_i|| = ||v_i - \mathrm{proj}_{V_{j-1}} v_i||$ which was the distance from v_i to V_{j-1}. The volume $\mathrm{vol}(E_j)$ of a j-dimensional parallelepiped E_j with base E_{j-1} in V_{j-1} and height $||u_i||$ is $\mathrm{vol}(E_{j-1})||u_j||$. Inductively $\mathrm{vol}(E_j) = ||u_j||\mathrm{vol}(E_{j-1})$ and therefore $\mathrm{vol}(E_n) = \prod_{j=1}^n ||u_j|| = \det(R)$.

The volume of a k dimensional parallelepiped defined by the vectors $v_1, \dots, v_k$ is $\sqrt{\det(A^T A)}$.

Proof. $Q^T Q = I_n$ gives $A^T A = (QR)^T(QR) = R^T Q^T Q R = R^T R$. So, $\det(R^T R) = \det(R)^2 = (\prod_{j=1}^k ||u_j||)^2$. (Note that A is a $n \times k$ matrix and that $A^T A = R^T R$ and R are $k \times k$ matrices.)

ORIENTATION. Determinants allow to **define** the orientation of n vectors in n-dimensional space. This is "handy" because there is no "right hand rule" in hyperspace ... To do so, define the matrix A with column vectors v_j and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if v_1 is the thumb, v_2 is the pointing finger and v_3 is the middle finger, then their orientation is positive.



CRAMER'S RULE. This is an explicit formula for the solution of $A\vec{x} = \vec{b}$. If A_i denotes the matrix, where the column $\vec{v}_i$ of A is replaced by $\vec{b}$, then

$$\boxed{x_i = \det(A_i)/\det(A)}$$

Proof. $\det(A_i) = \det([v_1, \dots, b, \dots, v_n] = \det([v_1, \dots, (Ax), \dots, v_n] = \det([v_1, \dots, \sum_i x_i v_i, \dots, v_n] = x_i \det([v_1, \dots, v_i, \dots, v_n]) = x_i \det(A)$

EXAMPLE. Solve the system $5x+3y = 8, 8x+5y = 2$ using Cramers rule. This linear system with $A = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. We get $x = \det \begin{bmatrix} 8 & 3 \\ 2 & 5 \end{bmatrix} = 34 y = \det \begin{bmatrix} 5 & 8 \\ 8 & 2 \end{bmatrix} = -54$.

GABRIEL CRAMER. (1704-1752), born in Geneva, Switzerland, he worked on geometry and analysis. Cramer used the rule named after him in a book "Introduction à l'analyse des lignes courbes algébraique", where he solved like this a system of equations with 5 unknowns. According to a short biography of Cramer by J.J O'Connor and E F Robertson, the rule had however been used already before by other mathematicians. Solving systems with Cramer's formulas is slower than by Gaussian elimination. The rule is still important. For example, if A or b depends on a parameter t, and we want to see how x depends on the parameter t one can find explicit formulas for $(d/dt)x_i(t)$.

THE INVERSE OF A MATRIX. Because the columns of A^{-1} are solutions of $A\vec{x} = \vec{e}_i$, where $\vec{e}_j$ are basis vectors, Cramers rule together with the Laplace expansion gives the formula:

$$\boxed{[A^{-1}]_{ij} = (-1)^{i+j}\det(A_{ji})/\det(A)}$$

$B_{ij} = (-1)^{i+j}\det(A_{ji})$ is called the **classical adjoint** or **adjugate** of A. **Note** the change $ij \to ji$. **Don't** confuse the classical adjoint with the **transpose** A^T which is sometimes also called the **adjoint**.

EXAMPLE. $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 4 \\ 6 & 0 & 7 \end{bmatrix}$ has $\det(A) = -17$ and we get $A^{-1} = \begin{bmatrix} 14 & -21 & 10 \\ -11 & 8 & -3 \\ -12 & 18 & -11 \end{bmatrix} /(-17)$:

$B_{11} = (-1)^{1+1}\det\begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix} = 14$. $B_{12} = (-1)^{1+2}\det\begin{bmatrix} 3 & 1 \\ 0 & 7 \end{bmatrix} = -21$. $B_{13} = (-1)^{1+3}\det\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = 10$.

$B_{21} = (-1)^{2+1}\det\begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix} = -11$. $B_{22} = (-1)^{2+2}\det\begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix} = 8$. $B_{23} = (-1)^{2+3}\det\begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} = -3$.

$B_{31} = (-1)^{3+1}\det\begin{bmatrix} 5 & 2 \\ 6 & 0 \end{bmatrix} = -12$. $B_{32} = (-1)^{3+2}\det\begin{bmatrix} 2 & 3 \\ 6 & 0 \end{bmatrix} = 18$. $B_{33} = (-1)^{3+3}\det\begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} = -11$.

THE ART OF CALCULATING DETERMINANTS. When confronted with a matrix, it is good to go through a checklist of methods to crack the determinant. Often, there are different possibilities to solve the problem, in many cases the solution is particularly simple using one method.

- Is it a upper or lower triangular matrix?
- Is it a partitioned matrix?
- Is it a product like $\det(A^{1000}) = \det(A)^{1000}$?
- Is the matrix known to be non invertible and so $\det(A) = 0$?
- Do you see duplicated columns or rows?
- Can you row reduce to a triangular case?
- Are there only a few nonzero patters?
- Try Laplace expansion with some row or column?
- Later: can we compute the eigenvalues of A?

EXAMPLES.

1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 5 & 5 & 5 & 5 & 4 \\ 1 & 3 & 2 & 7 & 4 \\ 3 & 2 & 8 & 4 & 9 \end{bmatrix}$ Try row reduction.

2) $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ Laplace expansion.

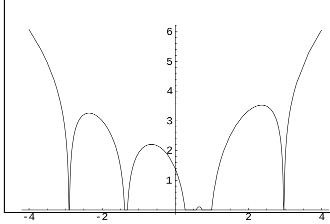
3) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$ Partitioned matrix.

4) $A = \begin{bmatrix} 1 & 6 & 10 & 1 & 15 \\ 2 & 8 & 17 & 1 & 29 \\ 0 & 0 & 3 & 8 & 12 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ Make it triangular.

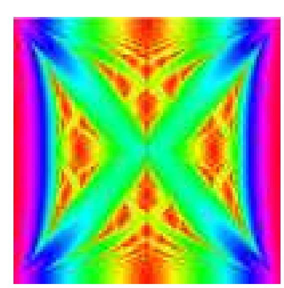
APPLICATION HOFSTADTER BUTTERFLY. In solid state physics, one is interested in the function $f(E) = \det(L - EI_n)$, where

$$L = \begin{bmatrix} \lambda\cos(\alpha) & 1 & 0 & \cdot & 0 & 1 \\ 1 & \lambda\cos(2\alpha) & 1 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & 1 & \lambda\cos((n-1)\alpha) & 1 \\ 1 & 0 & \cdot & 0 & 1 & \lambda\cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, E is the energy and $\alpha = 2\pi/n$. The electron can move as a Bloch wave whenever the determinant is negative. These intervals form the **spectrum** of the quantum mechanical system. A physicist is interested in the rate of change of $f(E)$ or its dependence on λ when E is fixed. .



The graph to the left shows the function $E \mapsto \log(|\det(L - EI_n)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator. The picture to the right shows the spectrum of the crystal depending on α. It is called the "Hofstadter butterfly" made popular in the book "Gödel, Escher Bach" by Douglas Hofstadter.



If we swap two rows, the determinant changes sign. If we scale a row by a factor λ, the determinant gets multiplied by λ. If two rows are the same or two columns are the same, the determinant is zero. Subtracting a row from an other does not change the determinant. This allows us to compute the determinant by row reduction. You only need to row reduce until it is triangular!

When confronted with a matrix, it is good to go through a checklist of methods to crack the determinant. Often, there are different possibilities to solve the problem, in many cases the solution is particularly simple using one method.

- Is it a upper or lower triangular matrix?
- Is it a partitioned matrix?
- Is it a product like $\det(A^{1000}) = \det(A)^{1000}$?
- Is the matrix known to be non invertible and so $\det(A) = 0$?
- Is the matrix orthogonal?
- Do you see duplicated columns or rows?
- Can you row reduce to a triangular case?
- Are there only a few nonzero patters?
- Try Laplace expansion with some row or column?
- Later: can we compute the eigenvalues of A?

EXAMPLES.

1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 5 & 5 & 5 & 5 & 4 \\ 1 & 3 & 2 & 7 & 4 \\ 3 & 2 & 8 & 4 & 9 \end{bmatrix}$ Try row reduction.

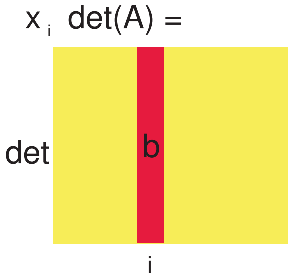
2) $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ Laplace expansion.

3) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$ Partitioned matrix.

4) $A = \begin{bmatrix} 1 & 6 & 10 & 1 & 15 \\ 2 & 8 & 17 & 1 & 29 \\ 0 & 0 & 3 & 8 & 12 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ Make it triangular.

here are more applications:

Determinants allow to **define** the orientation of n vectors in n-dimensional space. This is "handy" because there is no "right hand rule" in hyperspace ... To do so, define the matrix A with column vectors v_j and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if v_1 is the thumb, v_2 is the pointing finger and v_3 is the middle finger, then their orientation is positive.



An application This is an explicit formula for the solution of $A\vec{x} = \vec{b}$. If A_i denotes the matrix, where the column $\vec{v}_i$ of A is replaced by $\vec{b}$, then

$$\boxed{x_i = \det(A_i)/\det(A)}$$

Proof. $\det(A_i) = \det([v_1, \dots, b, \dots, v_n] = \det([v_1, \dots, (Ax), \dots, v_n] = \det([v_1, \dots, \sum_i x_i v_i, \dots, v_n] = x_i \det([v_1, \dots, v_i, \dots, v_n]) = x_i \det(A)$

EXAMPLE. Solve the system $5x + 3y = 8, 8x + 5y = 2$ using Cramers rule. This linear system with $A = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. We get $x = \det \begin{bmatrix} 8 & 3 \\ 2 & 5 \end{bmatrix} = 34 y = \det \begin{bmatrix} 5 & 8 \\ 8 & 2 \end{bmatrix} = -54$.

GABRIEL CRAMER. (1704-1752), born in Geneva, Switzerland, he worked on geometry and analysis. Cramer used the rule named after him in a book "Introduction à l'analyse des lignes courbes algébraique", where he solved like this a system of equations with 5 unknowns. According to a short biography of Cramer by J.J O'Connor and E F Robertson, the rule had however been used already before by other mathematicians. Solving systems with Cramer's formulas is slower than by Gaussian elimination. The rule is still important. For example, if A or b depends on a parameter t, and we want to see how x depends on the parameter t one can find explicit formulas for $(d/dt)x_i(t)$.

THE INVERSE OF A MATRIX. Because the columns of A^{-1} are solutions of $A\vec{x} = \vec{e}_i$, where $\vec{e}_j$ are basis vectors, Cramers rule together with the Laplace expansion gives the formula:

$$\boxed{[A^{-1}]_{ij} = (-1)^{i+j} \det(A_{ji})/\det(A)}$$

$B_{ij} = (-1)^{i+j} \det(A_{ji})$ is called the **classical adjoint** or **adjugate** of A. **Note** the change $ij \to ji$. **Don't** confuse the classical adjoint with the **transpose** A^T which is sometimes also called the **adjoint**.

EXAMPLE. $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 4 \\ 6 & 0 & 7 \end{bmatrix}$ has $\det(A) = -17$ and we get $A^{-1} = \begin{bmatrix} 14 & -21 & 10 \\ -11 & 8 & -3 \\ -12 & 18 & -11 \end{bmatrix} /(-17)$:

$B_{11} = (-1)^{1+1} \det \begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix} = 14$. $B_{12} = (-1)^{1+2} \det \begin{bmatrix} 3 & 1 \\ 0 & 7 \end{bmatrix} = -21$. $B_{13} = (-1)^{1+3} \det \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = 10$.

$B_{21} = (-1)^{2+1} \det \begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix} = -11$. $B_{22} = (-1)^{2+2} \det \begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix} = 8$. $B_{23} = (-1)^{2+3} \det \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} = -3$.

$B_{31} = (-1)^{3+1} \det \begin{bmatrix} 5 & 2 \\ 6 & 0 \end{bmatrix} = -12$. $B_{32} = (-1)^{3+2} \det \begin{bmatrix} 2 & 3 \\ 6 & 0 \end{bmatrix} = 18$. $B_{33} = (-1)^{3+3} \det \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} = -11$.

EIGENVALUES & DYNAMICAL SYSTEMS — 21b,O.Knill

EIGENVALUES AND EIGENVECTORS. A nonzero vector v is called an **eigenvector** of a $n \times n$ matrix A with **eigenvalue** λ if $\boxed{Av = \lambda v}$.

EXAMPLES.

- $\vec{v}$ is an eigenvector to the eigenvalue 0 if $\vec{v}$ is in the kernel of A.
- A rotation in space has an eigenvalue 1, with eigenvector spanning the axes of rotation.
- If A is a diagonal matrix with diagonal elements a_i, then $\vec{e}_i$ is an eigenvector with eigenvalue a_i.
- A shear A in the direction v has an eigenvector $\vec{v}$.
- Projections have eigenvalues 1 or 0.
- Reflections have eigenvalues 1 or -1.
- A rotation in the plane by an angle 30 degrees has no real eigenvector. (the eigenvectors are complex).

LINEAR DYNAMICAL SYSTEMS.
When applying a linear map $x \mapsto Ax$ again and again, obtain a **discrete dynamical system**. We want to understand what happens with the **orbit** $x_1 = Ax, x_2 = AAx = A^2x, x_3 = AAAx = A^3x, \ldots$.

EXAMPLE 1: $x \mapsto ax$ or $x_{n+1} = ax_n$ has the solution $x_n = a^n x_0$. For example, $1.03^{20} \cdot 1000 = 1806.11$ is the balance on a bank account which had 1000 dollars 20 years ago and if the interest rate was constant 3 percent.

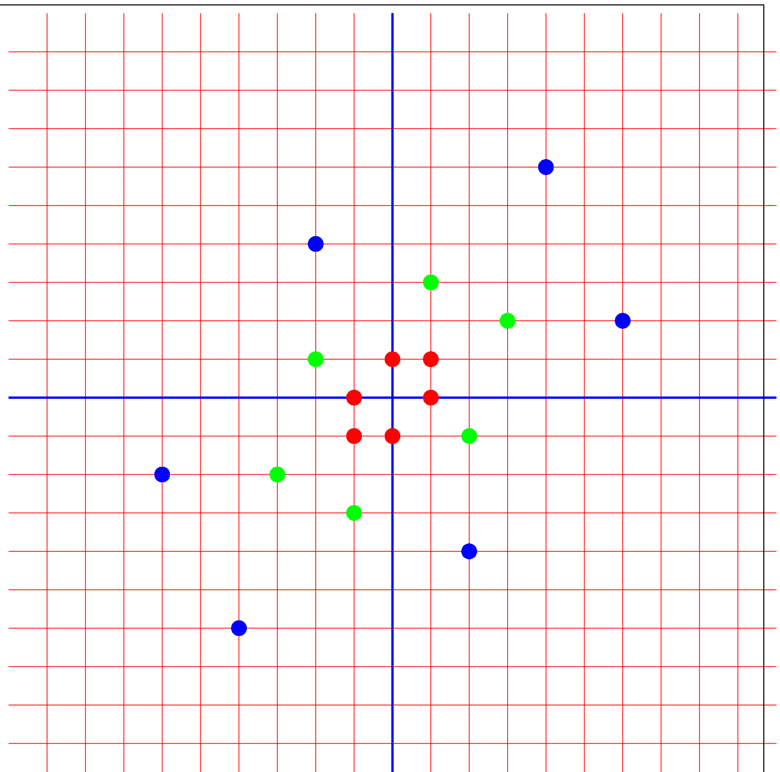
EXAMPLE 2: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $A\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A^2\vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. $A^3\vec{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. $A^4\vec{v} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ etc.

EXAMPLE 3: If $\vec{v}$ is an eigenvector with eigenvalue λ, then $A\vec{v} = \lambda\vec{v}$, $A^2\vec{v} = A(A\vec{v})) = A\lambda\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}$ and more generally $A^n\vec{v} = \lambda^n\vec{v}$.

RECURSION: If a scalar quantity u_{n+1} does not only depend on u_n but also on u_{n-1} we can write $(x_n, y_n) = (u_n, u_{n-1})$ and get a linear map because x_{n+1}, y_{n+1} depend in a linear way on x_n, y_n.

EXAMPLE: Lets look at the recursion $u_{n+1} = u_n - u_{n-1}$ with $u_0 = 0, u_1 = 1$. Because $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix}$. The recursion is done by iterating the matrix A: $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ $A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. We see that A^6 is the identity. Every initial vector is mapped after 6 iterations back to its original starting point.

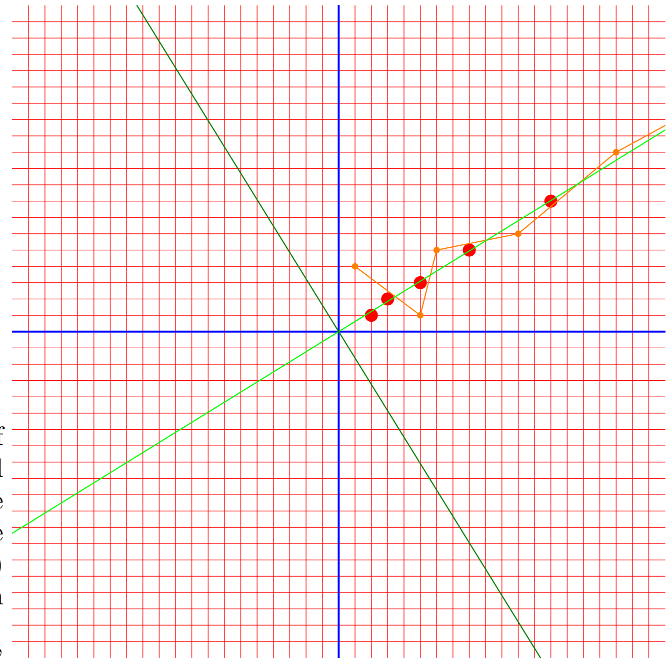
If the E parameter is changed, the dynamics also changes. For $E = 3$ for example, most initial points will escape to infinity similar as in the next example. Indeed, for $E = 3$, there is an eigenvector $\vec{v} = (3 + \sqrt{5})/2$ to the eigenvalue $\lambda = (3 + \sqrt{5})/2$ and $A^n\vec{v} = \lambda^n\vec{v}$ escapes to ∞.



THE FIBONACCI RECURSION:

In the third section of **Liber abbaci**, published in 1202, the mathematician Fibonacci, with real name **Leonardo di Pisa** (1170-1250) writes:

"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"



Mathematically, how does u_n grow, if $u_{n+1} = u_n + u_{n-1}$? We can assume $u_0 = 1$ and $u_1 = 2$ to match Leonardo's example. The sequence is $(1, 2, 3, 5, 8, 13, 21, ...)$. As before we can write this recursion using vectors $(x_n, y_n) = (u_n, u_{n-1})$ starting with $(1, 2)$. The matrix A to this recursion is $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Iterating gives $A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = A \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

ASSUME WE KNOW THE EIGENVALUES AND VECTORS: If $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$ and $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, we have an explicit solution $A^n\vec{v} = c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2$. This motivates to find good methods to compute eigenvalues and eigenvectors.

EVOLUTION OF QUANTITIES: Market systems, population quantities of different species, or ingredient quantities in a chemical reaction. A linear description might not always be be a good model but it has the advantage that we can solve the system explicitly. Eigenvectors will provide the key to do so.

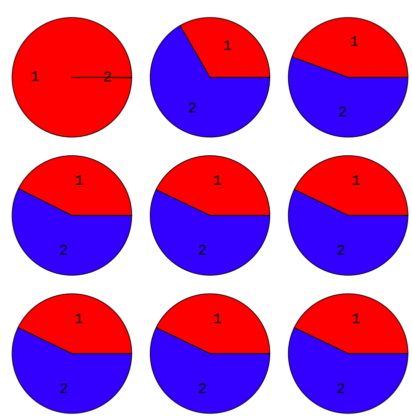
MARKOV MATRICES. A matrix with nonzero entries for which the sum of the columns entries add up to 1 is called a **Markov matrix**.

Markov Matrices have an eigenvalue 1.

Proof. The eigenvalues of A and A^T are the same because they have the same characteristic polynomial. The matrix A^T has an eigenvector $[1, 1, 1, 1, 1]^T$. An example is the matrix

$$A = \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 1/4 & 1/3 & 1/3 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

MARKOV PROCESS EXAMPLE: The percentage of people using Apple OS or the Linux OS is represented by a vector $\begin{bmatrix} m \\ l \end{bmatrix}$. Each cycle 2/3 of Mac OS users switch to Linux and 1/3 stays. Also lets assume that 1/2 of the Linux OS users switch to apple and 1/2 stay. The matrix $P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$ is a **Markov matrix**. What ratio of Apple/Linux users do we have after things settle to an equilibrium? We can simulate this with a dice: start in a state like $M = (1, 0)$ (all users have Macs). If the dice shows 3,4,5 or 6, a user in that group switch to Linux, otherwise stays in the M camp. Throw also a dice for each user in L. If 1,2 or 3 shows up, the user switches to M. The matrix P has an eigenvector $(3/7, 4/7)$ which belongs to the eigenvalue 1. The interpretation of $P\vec{v} = \vec{v}$ is that with this split up, there is no change in average.



Choose 4 random numbers between 1 and 6.

A	B	C	D

If the vector is e_1, then only go down if the number is 1 otherwise, go up. If the vector is e_2, then go down if the number is odd, otherwise, go up.
We now find experimentally the eigenvector of the Markov matrix

$$A = \begin{bmatrix} 5/6 & 1/2 \\ 1/6 & 1/2 \end{bmatrix}$$

to the eigenvalue 1 in class.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

COMPUTING EIGENVALUES — Math 21b, O.Knill

THE TRACE. The **trace** of a matrix A is the sum of its diagonal elements.

EXAMPLES. The trace of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ is $1+4+8 = 13$. The trace of a skew symmetric matrix A is zero because there are zeros in the diagonal. The trace of I_n is n.

CHARACTERISTIC POLYNOMIAL. The polynomial $f_A(\lambda) = \det(A - \lambda I_n)$ is called the **characteristic polynomial** of A.

EXAMPLE. The characteristic polynomial of the matrix A above is $p_A(\lambda) = -\lambda^3 + 13\lambda^2 + 15\lambda$.

$\boxed{\text{The eigenvalues of } A \text{ are the roots of the characteristic polynomial } f_A(\lambda).}$

Proof. If λ is an eigenvalue of A with eigenfunction $\vec{v}$, then $A - \lambda$ has $\vec{v}$ in the kernel and $A - \lambda$ is not invertible so that $f_A(\lambda) = \det(A - \lambda) = 0$.

The polynomial has the form

$$\boxed{f_A(\lambda) = (-\lambda)^n + \mathrm{tr}(A)(-\lambda)^{n-1} + \cdots + \det(A)}$$

THE 2x2 CASE. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a+d)/2\lambda + (ad - bc)$. The eigenvalues are $\lambda_\pm = T/2 \pm \sqrt{(T/2)^2 - D}$, where T is the trace and D is the determinant. In order that this is real, we must have $(T/2)^2 \geq D$.

EXAMPLE. The characteristic polynomial of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ is $\lambda^2 - 3\lambda + 2$ which has the roots $1, 2$: $f_A(\lambda) = (1 - \lambda)(2 - \lambda)$.

THE FIBONNACCI RABBITS. The Fibonacci's recursion $u_{n+1} = u_n + u_{n-1}$ defines the growth of the rabbit population. We have seen that it can be rewritten as $\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The roots of the characteristic polynomial $f_A(x) = \lambda^2 - \lambda - 1$ are $(\sqrt{5}+1)/2, (\sqrt{5}-1)/2$.

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$ then λ is said to be an eigenvalue of **algebraic multiplicity** k.

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2 and the eigenvalue $\lambda = 2$ with algebraic multiplicity 1.

HOW TO COMPUTE EIGENVECTORS? Because $(A - \lambda)v = 0$, the vector v is in the kernel of $A - \lambda$. We know how to compute the kernel.

EXAMPLE FIBONNACCI. The kernel of $A - \lambda I_2 = \begin{bmatrix} 1 - \lambda_\pm & 1 \\ 1 & 1 - \lambda_\pm \end{bmatrix}$ is spanned by $\vec{v}_+ = \begin{bmatrix} (1+\sqrt{5})/2, 1 \end{bmatrix}^T$ and $\vec{v}_- = \begin{bmatrix} (1-\sqrt{5})/2, 1 \end{bmatrix}^T$. They form a basis $\mathcal{B}$.

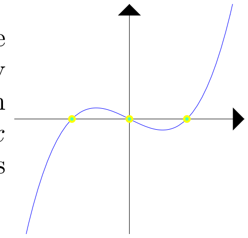
SOLUTION OF FIBONNACCI. To obtain a formula for $A^n\vec{v}$ with $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we form $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}/\sqrt{5}$.

Now, $\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A^n\vec{v} = A^n(\vec{v}_+/\sqrt{5} - \vec{v}_-/\sqrt{5}) = A^n\vec{v}_+/\sqrt{5} - A^n(\vec{v}_-/\sqrt{5} = \lambda_+^n\vec{v}_+/\sqrt{5} - \lambda_-^n\vec{v}_+/\sqrt{5}$. We see that $u_n = [(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]/\sqrt{5}$.

ROOTS OF POLYNOMIALS.

For polynomials of degree 3 and 4 there exist explicit formulas in terms of radicals. As Galois (1811-1832) and Abel (1802-1829) have shown, it is not possible for equations of degree 5 or higher. Still, one can compute the roots numerically.

REAL SOLUTIONS. A $(2n+1) \times (2n+1)$ matrix A always has a real eigenvalue because the characteristic polynomial $p(x) = x^5 + ... + \det(A)$ has the property that $p(x)$ goes to $\pm\infty$ for $x \to \pm\infty$. Because there exist values a, b for which $p(a) < 0$ and $p(b) > 0$, by the intermediate value theorem, there exists a real x with $p(x) = 0$. Application: A rotation in 11 dimensional space has all eigenvalues $|\lambda| = 1$. The real eigenvalue must have an eigenvalue 1 or -1.



EIGENVALUES OF TRANSPOSE. We know that the characteristic polynomials of A and the transpose A^T agree because $\det(B) = \det(B^T)$ for any matrix. Therefore A and A^T have the same eigenvalues.

APPLICATION: MARKOV MATRICES. A matrix A for which each column sums up to 1 is called a **Markov matrix**. The transpose of a Markov matrix has the eigenvector $\begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}$ with eigenvalue 1. Therefore:

$$\boxed{\text{A Markov matrix has an eigenvector } \vec{v} \text{ to the eigenvalue 1.}}$$

This vector $\vec{v}$ defines an equilibrium point of the Markov process.

EXAMPLE. If $A = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$. Then $[3/7, 4/7]$ is the equilibrium eigenvector to the eigenvalue 1.

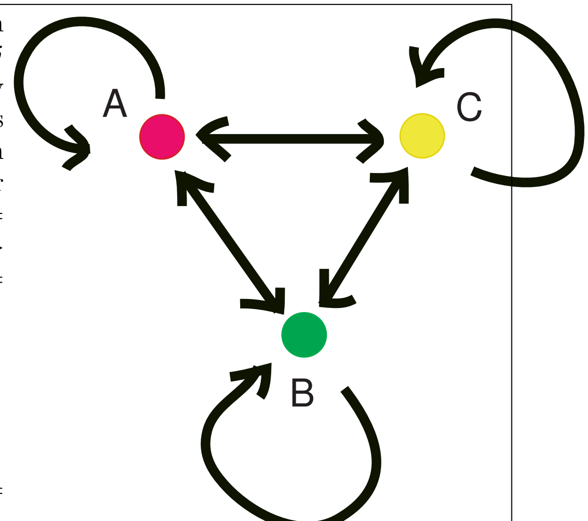
BRETSCHERS HOMETOWN. In Bretscher's book there is a Markov problem about Andelfingen, the hometown of Bretscher, where people shop in two shops. Initially all shop in shop W. After a new shop opens, every week 20 percent switch to the other shop M. Missing something at the new place, every week, 10 percent switch back. This leads to a Markov matrix $A = \begin{bmatrix} 8/10 & 1/10 \\ 2/10 & 9/10 \end{bmatrix}$. After some time, things will settle down and we will have certain percentage shopping in W and other percentage shopping in M. This is the equilibrium.

.

MARKOV PROCESS IN PROBABILITY. Assume we have a graph like a network and at each node i, the probability to go from i to j in the next step is $[A]_{ij}$, where A_{ij} is a Markov matrix. We know from the above result that there is an eigenvector $\vec{p}$ which satisfies $A\vec{p} = \vec{p}$. It can be normalized that $\sum_i p_i = 1$. The interpretation is that p_i is the probability that the walker is on the node p. For example, on a triangle, we can have the probabilities: $P(A \to B) = 1/2, P(A \to C) = 1/4, P(A \to A) = 1/4, P(B \to A) = 1/3, P(B \to B) = 1/6, P(B \to C) = 1/2, P(C \to A) = 1/2, P(C \to B) = 1/3, P(C \to C) = 1/6$. The corresponding matrix is

$$A = \begin{bmatrix} 1/4 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/4 & 1/2 & 1/6 \end{bmatrix} .$$

In this case, the eigenvector to the eigenvalue 1 is $p = [38/107, 36/107, 33/107]^T$.



CALCULATING EIGENVECTORS — Math 21b, O.Knill

NOTATION. We often just write 1 instead of the identity matrix I_n or λ instead of λI_n.

COMPUTING EIGENVALUES. Recall: because $A-\lambda$ has $\vec{v}$ in the kernel if λ is an eigenvalue the characteristic polynomial $f_A(\lambda) = \det(A-\lambda) = 0$ has eigenvalues as roots.

2×2 CASE. Recall: The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a+d)/2\lambda + (ad-bc)$. The eigenvalues are $\lambda_\pm = T/2 \pm \sqrt{(T/2)^2 - D}$, where $T = a+d$ is the trace and $D = ad-bc$ is the determinant of A. If $(T/2)^2 \geq D$, then the eigenvalues are real.

NUMBER OF ROOTS. Recall: There are examples with no real eigenvalue (i.e. rotations). By inspecting the graphs of the polynomials, one can deduce that $n\times n$ matrices with odd n always have a real eigenvalue. Also $n \times n$ matrices with even n and a negative determinant always have a real eigenvalue.

IF ALL ROOTS ARE REAL. $f_A(\lambda) = (-\lambda)^n + \mathrm{tr}(A)(-\lambda)^{n-1} + \cdots + \det(A) = (\lambda_1 - \lambda)\dots(\lambda_n - \lambda)$, we see that $\lambda_1 + \cdots + \lambda_n = \mathrm{trace}(A)$ and $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det(A)$.

HOW TO COMPUTE EIGENVECTORS? Because $(\lambda - A)\vec{v} = 0$, the vector $\vec{v}$ is in the kernel of $\lambda - A$.

EIGENVECTORS of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\vec{v}_\pm$ with eigenvalue $\lambda_\pm$.

If $c = d = 0$, then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors.

If $c \neq 0$, then the eigenvectors to $\lambda_\pm$ are $\begin{bmatrix} \lambda_\pm - d \\ c \end{bmatrix}$.

If $b \neq 0$, then the eigenvectors to $\lambda_\pm$ are $\begin{bmatrix} b \\ \lambda_\pm - d \end{bmatrix}$

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$, then f has **algebraic multiplicity** k. If A is similar to an upper triangular matrix B, then it is the number of times that λ_0 occurs in the diagonal of B.

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2 and eigenvalue 2 with algebraic multiplicity 1.

GEOMETRIC MULTIPLICITY. The dimension of the eigenspace E_λ of an eigenvalue λ is called the **geometric** multiplicity of λ.

EXAMPLE: the matrix of a shear is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has the eigenvalue 1 with algebraic multiplicity 2. The kernel of $A - 1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the geometric multiplicity is 1.

EXAMPLE: The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has eigenvalue 1 with algebraic multiplicity 2 and the eigenvalue 0 with multiplicity 1. Eigenvectors to the eigenvalue $\lambda = 1$ are in the kernel of $A - 1$ which is the kernel of $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The geometric multiplicity is 1.

RELATION BETWEEN ALGEBRAIC AND GEOMETRIC MULTIPLICITY.

The geometric multiplicity is smaller or equal than the algebraic multiplicity.

EXAMPLE. What are the algebraic and geometric multiplicities of $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$?

SOLUTION. The algebraic multiplicity of the eigenvalue 2 is 5. To get the kernel of $A-2$, one solves the system of equations $x_4 = x_3 = x_2 = x_1 = 0$ so that the geometric multiplicity of the eigenvalues 2 is 1.

CASE: ALL EIGENVALUES ARE DIFFERENT.

If all eigenvalues are different, then all eigenvectors are linearly independent and all geometric and algebraic multiplicities are 1.

PROOF. Let λ_i be an eigenvalue different from 0 and assume the eigenvectors are linearly dependent. We have $v_i = \sum_{j \neq i} a_j v_j$ and $\lambda_i v_i = A v_i = A(\sum_{j \neq i} a_j v_j) = \sum_{j \neq i} a_j \lambda_j v_j$ so that $v_i = \sum_{j \neq i} b_j v_j$ with $b_j = a_j \lambda_j / \lambda_i$. If the eigenvalues are different, then $a_j \neq b_j$ and by subtracting $v_i = \sum_{j \neq i} a_j v_j$ from $v_i = \sum_{j \neq i} b_j v_j$, we get $0 = \sum_{j \neq i} (b_j - a_j) v_j = 0$. Now $(n-1)$ eigenvectors of the n eigenvectors are linearly dependent. Use induction.

CONSEQUENCE. If all eigenvalues of a $n \times n$ matrix A are different, there is an **eigenbasis**, a basis consisting of eigenvectors.

EXAMPLES. 1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ has eigenvalues $1, 3$ to the eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

2) $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ has an eigenvalue 3 with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but no other eigenvector. We do not have a basis.

3) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, every vector is an eigenvector. The standard basis is an eigenbasis.

Here are some tricks:

1) If the matrix is upper triangular or lower triangular one can read off the eigenvalues at the diagonal. The eigenvalues can be computed fast because row reduction is easy.

2) For 2×2 matrices, one can immediately write down the eigenvalues and eigenvectors:

The eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are

$$\lambda_{\pm} = \frac{\mathrm{tr}(A) \pm \sqrt{(\mathrm{tr}(A))^2 - 4\det(A)}}{2}$$

The eigenvectors in the case $c \neq 0$ are

$$v_{\pm} = \begin{bmatrix} \lambda_{\pm} - d \\ c \end{bmatrix} .$$

If both b and c are zero, then the standard basis is the eigenbasis.

3) How do we construct 2x2 matrices which have integer eigenvectors and integer eigenvalues? Just take an integer matrix for which the row vectors have the same sum. Then this sum is an eigenvalue to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The other eigenvalue can be obtained by noticing that the trace of the matrix is the sum of the eigenvalues. For example, the matrix $\begin{bmatrix} 6 & 7 \\ 2 & 11 \end{bmatrix}$ has the eigenvalue 13 and because the sum of the eigenvalues is 18 a second eigenvalue 5.

4) If you see a partitioned matrix

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

then the union of the eigenvalues of A and B are the eigenvalues of C. If v is an eigenvector of A, then $\begin{bmatrix} v \\ 0 \end{bmatrix}$ is an eigenvector of C. If w is an eigenvector of B, then $\begin{bmatrix} 0 \\ w \end{bmatrix}$ is an eigenvector of C.

DIAGONALIZATION — Math 21b, O.Knill

If an $n \times n$ matrix have n linearly independent vectors, they form an **eigenbasis** and A is diagonalizable. When does a matrix A have an eigenbasis?

Assume we have an eigenbasis. If S is the matrix with the eigenvectors as columns, then $\boxed{B = S^{-1}AS}$ is diagonal.
Proof: We have $S\vec{e}_i = \vec{v}_i$ and $AS\vec{e}_i = \lambda_i\vec{v}_i$ we know $S^{-1}AS\vec{e}_i = \lambda_i\vec{e}_i$. Therefore, B is diagonal with diagonal entries λ_i.

$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ has the eigenvalues $\lambda_1 = 2+\sqrt{3}$ with eigenvector $\vec{v}_1 = [\sqrt{3}, 1]$ and the eigenvalues $\lambda_2 = 2 - \sqrt{3}$. with eigenvector $\vec{v}_2 = [-\sqrt{3}, 1]$. Form $S = \begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{bmatrix}$ and check $S^{-1}AS = D$ is diagonal.

Theorem: similar matrices have the same eigenvalues.

One can see this in two ways:

1) If $B = S^{-1}AS$ and $\vec{v}$ is an eigenvector of B to the eigenvalue λ, then $S\vec{v}$ is an eigenvector of A to the eigenvalue λ.

2) From $\det(S^{-1}AS) = \det(A)$, we know that the characteristic polynomials $f_B(\lambda) = \det(\lambda - B) = \det(\lambda - S^{-1}AS) = \det(S^{-1}(\lambda - AS) = \det((\lambda - A) = f_A(\lambda)$ are the same.

Theorem: similar matrices have the same geometric multiplicities.
$Bv = \lambda v$ implies $S^{-1}ASv = \lambda v$ so that $ASv = \lambda Sv$. We see that S maps eigenvectors of B to eigenvectors of A. If k vectors are linearly independent in $E_{B,\lambda}$ then the k vectors are linearly independent in $E_{A,\lambda}$.

Because the characteristic polynomials of similar matrices agree, the trace $\mathrm{tr}(A)$, the determinant and the **eigenvalues** of similar matrices agree. We can use this to find out, whether two matrices are similar.

Criteria for similarity:

If A and B have the same characteristic polynomial and diagonalizable, then they are similar.
If A and B have a different determinant or trace, they are not similar.
If A has an eigenvalue which is not an eigenvalue of B, then they are not similar.
If A and B have the same eigenvalues but different geometric multiplicities, then they are not similar.

It is possible to give an "if and only if" condition for similarity even so this is usually not covered or only referred to by more difficult theorems which uses also the power trick we have used before:

If A and B have the same eigenvalues and the corresponding geometric multiplicities which agree and the same holds for all powers A^k and B^k, then A is similar to B.

The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

has eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

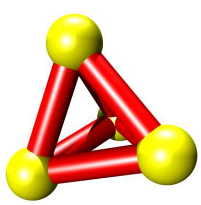
and eigenvalues $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_3 = 4$. It is diagonalizable even so we have multiple eigenvalues. With $S = [v_1 v_2 v_3 v_4]$, the matrix $B = S^{-1}BS$ is diagonal with entries $0, 0, 0, 4$.

An important theorem:

If all eigenvalues of a matrix A are different, then the matrix A is diagonalizable.

Why do we want to diagonalize?

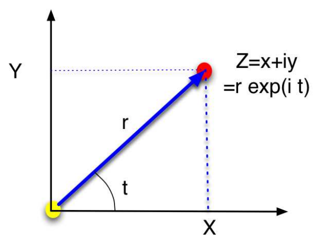
- It clarifies discrete dynamical systems.
- Solve differential equations.
- We can define functions of matrices like $p(A)$ with $p(x) = 1 + x + x^2 + x^3/6$.



The white phosphorus is a molecule of tetrahedral form. The Laplacian of this matrix is

$$A = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

Find the eigenvalues and eigenvectors. But the eigenvalues are are the eigenvectors to the eigenvalues $\lambda_1 = 0, \lambda_2 = -4, \lambda_3 = -4, \lambda_3 = -4$. With $S = [v_1 v_2 v_3 v_4]$, the matrix $B = S^{-1}BS$ is diagonal.



Complex numbers were associated to rotation dilation matrices already. We write complex numbers as $z = x + iy$ or $r\exp(i\phi) = r\cos(\phi) + ir\sin(\phi)$. The real number $r = |z|$ is the **absolute value** of z and ϕ is the **argument** and denoted by $\arg(z)$. Complex numbers contain the **real numbers** $z = x + i0$ as a subset. One writes $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$ if $z = x + iy$.

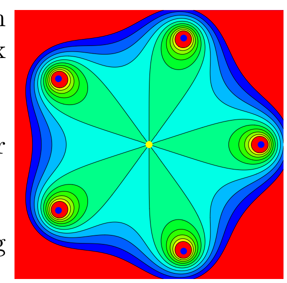
Complex numbers are added like vectors: $x + iy + u + iv = (x + u) + i(y + v)$ and multiplied as $z * w = (x + iy)(u + iv) = xu - yv + i(yu - xv)$. If $z \neq 0$, one can divide $1/z = 1/(x + iy) = (x - iy)/(x^2 + y^2)$.

The absolute value $|z| = \sqrt{x^2 + y^2}$ satisfies $|zw| = |z|\ |w|$. The argument satisfies $\arg(zw) = \arg(z) + \arg(w)$. These are direct consequences of the polar representation $z = r\exp(i\phi), w = s\exp(i\psi), zw = rs\exp(i(\phi + \psi))$.

If $z = x + iy$ is written as a vector $\begin{bmatrix} x \\ y \end{bmatrix}$, then multiplication with an other complex number w is a **dilation-rotation**: a scaling by $|w|$ and a rotation by $\arg(w)$.

$z^n = \exp(in\phi) = \cos(n\phi) + i\sin(n\phi) = (\cos(\phi) + i\sin(\phi))^n$ follows directly from $z = \exp(i\phi)$ but it is magic: it leads for example to formulas like $\cos(3\phi) = \cos(\phi)^3 - 3\cos(\phi)\sin^2(\phi)$ which would be more difficult to come by using geometrical or power series arguments. This de Moivre formula is useful for example in integration problems like $\int \cos(x)^3\, dx$, which can be solved by using the above.

Complex numbers of length 1 have the form $z = \exp(i\phi)$ and are located on the **unit circle**. The characteristic polynomial $f_A(\lambda) = \lambda^5 - 1$ of the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ has all roots on the unit circle. The roots $\exp(2\pi ki/5)$, for $k = 0, \dots, 4$ are located on the unit circle. One can solve $z^5 = 1$ by rewriting it as $z^5 = e^{2\pi ik}$ and then take the 5'th root on both sides.



$\log(z)$ is defined for $z \neq 0$ as $\log|z| + i\arg(z)$. For example, $\log(2i) = \log(2) + i\pi/2$. Example: what is $(-1)^i$? ($(-1)^i = e^{-\pi}$. The logarithm is not defined at 0 and the imaginary part is define only up to 2π. For example, both $i\pi/2$ and $5i\pi/2$ can be answers to $\log(i)$ One usually limits to the principal branch, where $0 \leq \arg(z) < 2\pi$.

Historically, the struggle with $\sqrt{-1}$ is interesting. Nagging questions appeared for example when trying to find closed solutions for roots of polynomials. Cardano (1501-1576) was one of the first mathematicians who at least considered complex numbers but called them arithmetic subtleties which were "as refined as useless". With Bombelli (1526-1573), complex numbers found some practical use. It was Descartes (1596-1650) who called the roots of negative numbers "imaginary". Although the fundamental theorem of algebra) was still not proved in the 18th century, and complex numbers were not fully understood, the square root of minus one $\sqrt{-1}$ was used more and more. Euler (1707-1783) made the observation that $\exp(ix) = \cos x + i\sin x$ which has as a special case the **magic formula** $e^{i\pi} + 1 = 0$ which relate the constants $0, 1, \pi, e$ in one equation.

Early on, many mathematicians still dismissed complex numbers. Others used complex numbers extensively in their work. In 1620, Girard suggested that an equation may have as many roots as its degree in 1620. Leibniz (1646-1716) spent quite a bit of time trying to apply the laws of algebra to complex numbers. He and Johann Bernoulli used imaginary numbers as integration aids. Lambert used complex numbers for map projections, d'Alembert used them in hydrodynamics, while Euler, D'Alembert and Lagrange used them in incorrect proofs of the fundamental theorem of algebra. Leonard Euler was the first to use the symbol i for $\sqrt{-1}$.

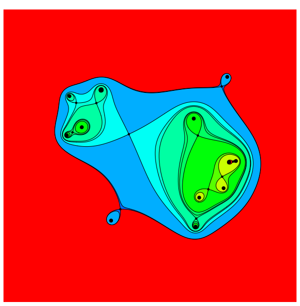
Gauss published the first correct proof of the fundamental theorem of algebra in his doctoral thesis, but still claimed in 1825 that **the true metaphysics of the square root of -1 is elusive** as late as 1825. By 1831 Gauss overcame his uncertainty about complex numbers and published his work on the geometric representation of complex numbers as points in the plane. In 1797, a Norwegian Caspar Wessel (1745-1818) and in 1806 a Swiss clerk named Jean Robert Argand (1768-1822) (who stated the theorem the first time for polynomials with complex coefficients) did similar work. But these efforts went unnoticed. William Rowan Hamilton (1805-1865) (who would also discover the quaternions while walking over a bridge) expressed in 1833 complex numbers as vectors.

Complex numbers continued to develop to **complex function theory** or **chaos theory**, a branch of dynamical systems theory. Complex numbers are helpful in geometry in number theory or in quantum mechanics. Once believed fictitious they are now most "natural numbers" and the "natural numbers" themselves are in fact the most "complex". A philosopher who asks "does $\sqrt{-1}$ really exist?" might be shown the representation of $x+iy$ as the rotation-dilation matrix $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. When adding or multiplying such dilation-rotation matrices, they behave like complex numbers: for example $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ plays the role of i.

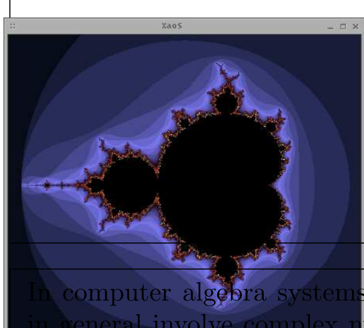
Fundamental theorem of algebra: (Gauss 1799) A polynomial of degree n has exactly n complex roots. It has a factorization $p(z) = c\prod(z_j - z)$ where z_j are the roots.

For us is important: a $n \times n$ matrix has n eigenvalues.
The characteristic polynomial $f_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$ satisfies $f_A(\lambda) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda)$, where λ_i are the roots of f.

Comparing $f_A(\lambda) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$ with $\lambda^n - \mathrm{tr}(A) + \ldots + (-1)^n \det(A)$ gives in full generality $\boxed{\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n,\ \det(A) = \lambda_1 \cdots \lambda_n.}$



The characteristic polynomial is an example of a function f from $\mathbf{C}$ to $\mathbf{C}$. The graph of this function would live in $\mathbb{C} \times \mathbb{C}$ which corresponds to a four dimensional real space. One can visualize the function however with the real-valued function $z \mapsto |f(z)|$. The figure to the left shows the contour lines of such a function $z \mapsto |f(z)|$, where f is a polynomial.



One would make a course about **complex dynamics**, the study of the the iteration of polynomials like $f_c(z) = z^2 + c$. The set of parameter values c for which the iterates $f_c(0), f_c^2(0) = f_c(f_c(0)), \ldots, f_c^n(0)$ stay bounded is called the **Mandelbrot set**. It is the fractal black region in the picture to the left. The Mandelbrot set is still under investigation. One still does not understand completely its topology.

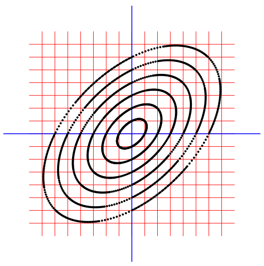
In computer algebra systems, the letter I is used for $i = \sqrt{-1}$. Eigenvalues or eigenvectors of a matrix will in general involve complex numbers. For example, in Mathematica, Eigenvalues[A] gives the eigenvalues of a matrix A and Eigensystem[A] gives the eigenvalues and the corresponding eigenvectors.

The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ has the characteristic polynomial $\lambda^2 - 2\cos(\phi) + 1$. The eigenvalues are $\cos(\phi) \pm \sqrt{\cos^2(\phi) - 1} = \cos(\phi) \pm i\sin(\phi) = \exp(\pm i\phi)$. The eigenvector to $\lambda_1 = \exp(i\phi)$ is $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and the eigenvector to the eigenvector $\lambda_2 = \exp(-i\phi)$ is $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

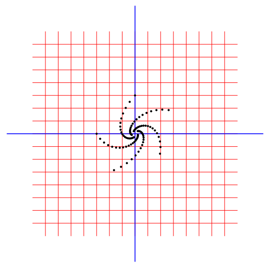
STABILITY **Math 21b, O. Knill**

A linear map $x \mapsto Ax$ defines a **dynamical system**. Iterating the linear map produces an **orbit** $x_0, x_1 = Ax, x_2 = A^2 = AAx, ...$ The vector $x_n = A^n x_0$ describes the situation of the system at **time** n.

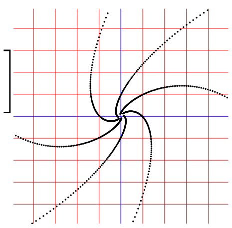
What happens with x_n when time evolves? Can one describe what happens asymptotically when time n goes to infinity?
In the case of the Fibonacci sequence x_n which gives the number of rabbits in a rabbit population at time n, the population grows exponentially. Such a behavior is called **unstable**. On the other hand, if A is a rotation, then $A^n \vec{v}$ stays bounded which is a type of **stability**. If A is a dilation with a dilation factor < 1, then $A^n \vec{v} \to 0$ for all $\vec{v}$, a thing which we will call **asymptotic stability**. The next pictures show experiments with some **orbits** $A^n \vec{v}$ with different matrices.



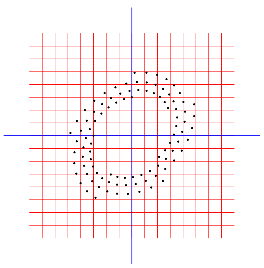
$\begin{bmatrix} 0.99 & -1 \\ 1 & 0 \end{bmatrix}$ stable (not asymptotic)



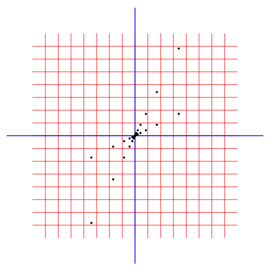
$\begin{bmatrix} 0.54 & -1 \\ 0.95 & 0 \end{bmatrix}$ asymptotic stable



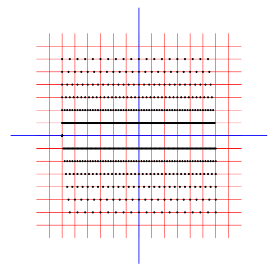
$\begin{bmatrix} 0.99 & -1 \\ 0.99 & 0 \end{bmatrix}$ asymptotic stable



$\begin{bmatrix} 0.54 & -1 \\ 1.01 & 0 \end{bmatrix}$ unstable



$\begin{bmatrix} 2.5 & -1 \\ 1 & 0 \end{bmatrix}$ unstable



$\begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$ unstable

The origin $\vec{0}$ is invariant under a linear map $T(\vec{x}) = A\vec{x}$. It is called **asymptotically stable** if $A^n(\vec{x}) \to \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$.

Let $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ be a dilation rotation matrix. Because multiplication with such a matrix is analogue to the multiplication with a complex number $z = p + iq$, the matrix A^n corresponds to a multiplication with $(p + iq)^n$. Since $|(p + iq)|^n = |p + iq|^n$, the origin is asymptotically stable if and only if $|p + iq| < 1$. Because $\det(A) = |p+iq|^2 = |z|^2$, rotation-dilation matrices A have an asymptotic stable origin if and only if $|\det(A)| < 1$.
Dilation-rotation matrices $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ have eigenvalues $p \pm iq$ and can be diagonalized in the complex.

If a matrix A has an eigenvalue $|\lambda| \geq 1$ to an eigenvector $\vec{v}$, then $A^n \vec{v} = \lambda^n \vec{v}$, whose length is $|\lambda^n|$ times the length of $\vec{v}$. So, we have no asymptotic stability if an eigenvalue satisfies $|\lambda| \geq 1$.

Sometimes we abbreviate "stable" for "asymptotically stable" even-so the commonly used term "stable" also includes linear maps like rotations, reflections or the identity. It is therefore preferable to leave the attribute "asymptotic".

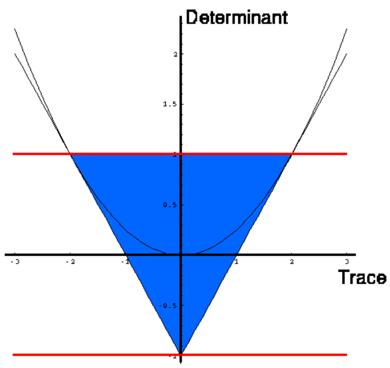
Rotations $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$ have the eigenvalue $\exp(\pm i\phi) = \cos(\phi) + i\sin(\phi)$ and are not asymptotically stable.

Dilations $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ have the eigenvalue r with algebraic and geometric multiplicity 2. Dilations are asymptotically stable if $|r| < 1$.

$\boxed{\text{A linear dynamical system } x \mapsto Ax \text{ has an asymptotically stable origin if and only if all its eigenvalues have an absolute value} < 1.}$

We have already seen in Example 3, that if one eigenvalue satisfies $|\lambda| > 1$, then the origin is not asymptotically stable. If $|\lambda_i| < 1$ for all i and all eigenvalues are different, there is an eigenbasis $v_1, \dots, v_n$. Every x can be written as $x = \sum_{j=1}^n x_j v_j$. Then, $A^n x = A^n(\sum_{j=1}^n x_j v_j) = \sum_{j=1}^n x_j \lambda_j^n v_j$ and because $|\lambda_j|^n \to 0$, there is stability. The proof of the general (non-diagonalizable) case reduces to the analysis of shear dilations.

The characteristic polynomial of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$. If $c \neq 0$, the eigenvalues are $\lambda_\pm = \mathrm{tr}(A)/2 \pm \sqrt{(\mathrm{tr}(A)/2)^2 - \det(A)}$. If the **discriminant** $(\mathrm{tr}(A)/2)^2 - \det(A)$ is non-negative, then the eigenvalues are real. This happens below the parabola, where the discriminant is zero.



In two dimensions, we have asymptotic stability if and only if $(\mathrm{tr}(A), \det(A))$ is contained in the interior of the **stability triangle** bounded by the lines $\det(A) = 1, \det(A) = \mathrm{tr}(A) - 1$ and $\det(A) = -\mathrm{tr}(A) - 1$.

Write $T = \mathrm{tr}(A)/2, D = \det(A)$. If $|D| \geq 1$, there is no asymptotic stability. If $\lambda = T + \sqrt{T^2 - D} = \pm 1$, then $T^2 - D = (\pm 1 - T)^2$ and $D = 1 \pm 2T$. For $D \leq -1 + |2T|$ we have a real eigenvalue ≥ 1. The conditions for stability is therefore $D > |2T| - 1$. It implies automatically $D > -1$ so that the triangle can be described shortly as $\boxed{|\mathrm{tr}(A)| - 1 < \det(A) < 1}$.

EXAMPLES.

1) The matrix $A = \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix}$ has determinant $5/4$ and trace 2 and the origin is unstable. It is a dilation-rotation matrix which corresponds to the complex number $1 + i/2$ which has an absolute value > 1.

2) A rotation A is never asymptotically stable: $\det(A)1$ and $\mathrm{tr}(A) = 2\cos(\phi)$. Rotations are the upper side of the **stability triangle**.

3) A dilation is asymptotically stable if and only if the scaling factor has norm < 1.

4) If $\det(A) = 1$ and $\mathrm{tr}(A) < 2$ then the eigenvalues are on the unit circle and there is no asymptotic stability.

5) If $\det(A) = -1$ (like for example Fibonacci) there is no asymptotic stability. For $\mathrm{tr}(A) = 0$, we are a corner of the stability triangle and the map is a reflection, which is not asymptotically stable neither.

SOME PROBLEMS.

1) If A is a matrix with asymptotically stable origin, what is the stability of 0 with respect to A^T?

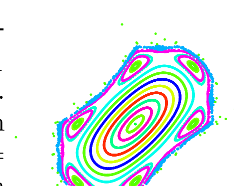
2) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to to A^{-1}?

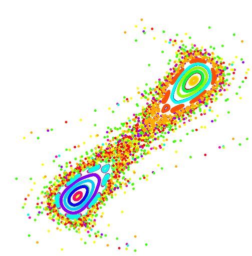
3) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to to A^{100}?

ON THE STABILITY QUESTION.

For general dynamical systems, the question of stability can be very difficult. We deal here only with linear dynamical systems, where the eigenvalues determine everything. For nonlinear systems, the story is not so simple even for simple maps like the Henon map. The questions go deeper: it is for example not known, whether our solar system is stable. We don't know whether in some future, one of the planets could get expelled from the solar system (this is a mathematical question because the escape time would be larger than the life time of the sun). For other dynamical systems like the atmosphere of the earth or the stock market, we would really like to know what happens in the near future ...

A pioneer in stability theory was Aleksandr Lyapunov (1857-1918). For nonlinear systems like $x_{n+1} = gx_n - x_n^3 - x_{n-1}$ the stability of the origin is nontrivial. As with Fibonacci, this can be written as $(x_{n+1}, x_n) = (gx_n - x_n^2 - x_{n-1}, x_n) = A(x_n, x_{n-1})$ called **cubic Henon map** in the plane. To the right are orbits in the cases $g = 1.5$, $g = 2.5$.





The first case is stable (but proving this requires a fancy theory called KAM theory), the second case is unstable (in this case actually the linearization at $\vec{0}$ determines the picture).

STABILE OR NOT? — Math 21b, O. Knill

FINANCE: INTEREST RATE.

$$A(x) = 1.04 \cdot x$$

BIOLOGY: POPULATION GROWTH.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Fibonnacci recursion $x_{n+1} = x_n + x_{n-1}$.

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

Lila Bush growth.

PHYSICS: SOLID STATE PHYSICS.

$$A = \begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix}$$

solves time independent Schrödinger equation $Hu = Eu$ which is $u_{n+1} + u_{n-1} = Eu_n$. Is this ever stable for any E?

STATISTICS: MARKOV PROCESSES.

$$A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Regular transition matrix for Wipf/Migros competion.

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$$

MCI/ATT/Sprint customers.

ECOLOGY: DIFFUSION.

$$A = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

Silvaplana, Sils, St Moritz lakes.

GEOMETRY: TRANSFORMATIONS.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

shear.

$$A = 1/2 \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$$

rotation dilation.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

projection

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

reflection

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

scaling transformation

THE SPECTRAL THEOREM Math 21b, O. Knill

A real matrix A is **symmetric**, if $A^T = A$.

Examples: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is symmetric, $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ is not symmetric.

Theorem: $\boxed{\text{Symmetric matrices } A \text{ have real eigenvalues.}}$

Proof: the dot product can be extended to complex vectors as $(v, w) = \sum_i \overline{v}_i w_i$. For real vectors it satisfies $(v, w) = v \cdot w$ and has the property $(Av, w) = (v, A^T w)$ for real matrices A and $(\lambda v, w) = \overline{\lambda}(v, w)$ as well as $(v, \lambda w) = \lambda(v, w)$. Now $\overline{\lambda}(v, v) = (\lambda v, v) = (Av, v) = (v, A^T v) = (v, Av) = (v, \lambda v) = \lambda(v, v)$ shows that $\overline{\lambda} = \lambda$ because $(v, v) \neq 0$ for $v \neq 0$.

Examples: The rotation dilation $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ has eigenvalues $a + ib$ which are real if and only if $b = 0$.
Reflection dilation matrices $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ are always symmetric.

$\boxed{\text{Eigenvectors to different eigenvalues are orthogonal!}}$

Proof. Assume $Av = \lambda v$ and $Aw = \mu w$. The relation $\lambda(v, w) = (\lambda v, w) = (Av, w) = (v, A^T w) = (v, Aw) = (v, \mu w) = \mu(v, w)$ is only possible if $(v, w) = 0$ if $\lambda \neq \mu$.

In applications, matrices are often symmetric. For example in **geometry** as **generalized dot products** $v \cdot Av$, or in **statistics** as **correlation matrices** $\mathrm{Cov}[X_k, X_l]$ or in quantum mechanics as **observables** or in **neural networks** as **learning maps** $x \mapsto \mathrm{sign}(Wx)$ or in graph theory as **adjacency matrices** etc. etc. Symmetric matrices play the same role as real numbers do among the complex numbers. Their eigenvalues often have physical or geometrical interpretations. One can also calculate with symmetric matrices like with numbers: for example, we can solve $B^2 = A$ for B if A is symmetric matrix and B is square root of A.) This is not possible in general: try to find a matrix B such that $B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$...

We have seen that if an eigenbasis exists, a matrix A is similar to a diagonal matrix $B = S^{-1}AS$, where $S = [v_1, \dots, v_n]$ is the matrix containing the basis vectors as columns. Similar matrices have the same characteristic polynomial $\det(B - \lambda) = \det(S^{-1}(A - \lambda)S) = \det(A - \lambda)$ and have therefore the same determinant, trace and eigenvalues. Physicists call the set of eigenvalues also **the spectrum** and say that similar matrices are isospectral. The spectrum is what you "see" (etymologically the name origins from the fact that in quantum mechanics the spectrum of radiation can be associated with eigenvalues of matrices.)

Spectral Theorem. $\boxed{\text{Symmetric matrices } A \text{ can be diagonalized } B = S^{-1}AS \text{ with an orthogonal } S.}$

PROOF. If all eigenvalues are different, there is an eigenbasis allowing diagonalization. The eigenvectors are all orthogonal and $B = S^{-1}AS$ contains the eigenvalues in the diagonal. In general, change the matrix A to $A = A + (C - A)t$ where C is a matrix with pairwise different eigenvalues. Then the eigenvalues are different for all except finitely many t. The orthogonal matrices S_t converges for $t \to 0$ to an orthogonal matrix S and S diagonalizes A.

WHY...? Why can we not perturb a general matrix A_t to have disjoint eigenvalues and A_t could be diagonalized: $S_t^{-1}A_tS_t = B_t$? The problem is that S_t might become singular for $t \to 0$.

Example: The matrix $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ has the eigenvalues $a+b, a-b$ and the eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}/\sqrt{2}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}/\sqrt{2}$. They are orthogonal. The orthogonal matrix $S = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ diagonalized A.

Example: The 3×3 matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has 2 eigenvalues 0 to the eigenvectors $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$ and one eigenvalue 3 to the eigenvector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. All these vectors can be made orthogonal and a diagonalization is possible even so the eigenvalues have multiplicities.

Example. The matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{bmatrix}$$

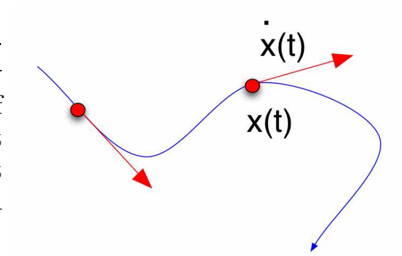
is symmetric. If we look at $B = A - 2I$, then we see that it has a 5 dimensional kernel. There are therefore 5 eigenvalues 0 of the matrix B. Furthermore, there is an eigenvalue 6 by the trace. Now the matrix A has eigenvalues $2, 2, 2, 2, 2, 8$. We have seen that we could compute all the eigenvalues pretty easily. An eigenbasis is

$$\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \} .$$

See see that as the theorem tells, the eigenvector to the eigenvalue 8 is perpendicular to the other. But the eigenvalues of the others are not. But we can use Gram-Schmidt to produce an orthonormal eigenbasis.

How do we find a square root of a given symmetric matrix? Because $S^{-1}AS = B$ is diagonal and we know how to take a square root of the diagonal matrix B, we can form $C = S\sqrt{B}S^{-1}$ which satisfies $C^2 = S\sqrt{B}S^{-1}S\sqrt{B}S^{-1} = SBS^{-1} = A$.

A differential equation $\frac{d}{dt}\vec{x} = f(\vec{x})$ defines a dynamical system. The solutions is a curve $\vec{x}(t)$ which has the **velocity vector** $f(\vec{x}(t))$ for all t. One often writes $\dot{x}$ instead of $\frac{d}{dt}x(t)$. We know a formula for the tangent at each point and aim to find a curve $\vec{x}(t)$ which starts at a given point $\vec{v} = \vec{x}(0)$ and has the prescribed direction and speed at each time t.



A system $\dot{x} = g(x,t)$ is the general differential equation in one dimensions. Examples are:

- If $\boxed{\dot{x} = g(t)}$, then $x(t) = \int_0^t g(t)\,dt$. Example: $\dot{x} = \sin(t), x(0) = 0$ has the solution $x(t) = \cos(t) - 1$.
- If $\boxed{\dot{x} = h(x)}$, then $dx/h(x) = dt$ and so $t = \int_0^x dx/h(x) = H(x)$ so that $x(t) = H^{-1}(t)$. Example: $\dot{x} = \frac{1}{\cos(x)}$ with $x(0) = 0$ gives $dx\cos(x) = dt$ and after integration $\sin(x) = t + C$ so that $x(t) = \arcsin(t + C)$. From $x(0) = 0$ we get $C = \pi/2$.
- If $\boxed{\dot{x} = g(t)/h(x)}$, then $H(x) = \int_0^x h(x)\,dx = \int_0^t g(t)\,dt = G(t)$ so that $x(t) = H^{-1}(G(t))$. Example: $\dot{x} = \sin(t)/x^2, x(0) = 0$ gives $dx x^2 = \sin(t)dt$ and after integration $x^3/3 = -\cos(t) + C$ so that $x(t) = (3C - 3\cos(t))^{1/3}$. From $x(0) = 0$ we obtain $C = 1$.

Remarks:
1) In general, we have no closed form solutions in terms of known functions. The solution $x(t) = \int_0^t e^{-t^2}\,dt$ of $\dot{x} = e^{-t^2}$ for example can not be expressed in terms of functions $\exp, \sin, \log, \sqrt{\cdot}$ etc but it can be solved using Taylor series: because $e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \dots$ taking coefficient wise the anti-derivatives gives: $x(t) = t - t^3/3 + t^4/(32!) - t^7/(73!) + \dots$.
2) The system $\dot{x} = g(x,t)$ can be written in the form $\vec{x} = f(\vec{x})$ with $\vec{x} = (x,t)$. $\frac{d}{dt}\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} g(x,t) \\ 1 \end{bmatrix}$.

The most general linear system in one dimension is $\boxed{\dot{x} = \lambda x}$. It has the solution $\boxed{x(t) = e^{\lambda t}x(0)}$. This differential equation appears

- as **population models** with $\lambda > 0$. The birth rate of the population is proportional to its size.
- as **radioactive decay** with $\lambda < 0$. The decay rate is proportional to the number of atoms.

Linear differential equations.

Linear dynamical systems have the form

$$\dot{x} = Ax$$

where A is a matrix and x is a vector, which depends on time t.

The origin $\vec{0}$ is always an **equilibrium point**: if $\vec{x}(0) = \vec{0}$, then $\vec{x}(t) = \vec{0}$ for all t. In general, we look for a solution $\vec{x}(t)$ for a given initial point $\vec{x}(0) = \vec{v}$. Here are three different ways to get a closed form solution:

- If $B = S^{-1}AS$ is diagonal with the eigenvalues $\lambda_j = a_j + ib_j$, then $y = S^{-1}x$ satisfies $y(t) = e^{Bt}$ and therefore $\boxed{y_j(t) = e^{\lambda_j t}y_j(0) = e^{a_j t}e^{ib_j t}y_j(0)}$. The solutions in the original coordinates are $x(t) = Sy(t)$.
- If $\vec{v}_i$ are the eigenvectors to the eigenvalues λ_i, and $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, then $\boxed{\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\vec{v}_n}$ is a closed formula for the solution of $\frac{d}{dt}\vec{x} = A\vec{x}, \vec{x}(0) = \vec{v}$.
- Linear differential equations can also be solved as in one dimensions: the general solution of $\dot{x} = Ax, \vec{x}(0) = \vec{v}$ is $\boxed{x(t) = e^{At}\vec{v} = (1 + At + A^2t^2/2! + \ldots)\vec{v}}$, because $\dot{x}(t) = A + 2A^2t/2! + \ldots = A(1 + At + A^2t^2/2! + \ldots)\vec{v} = Ae^{At}\vec{v} = Ax(t)$. This solution does not provide us with much insight however and this is why we prefer the closed form solution.

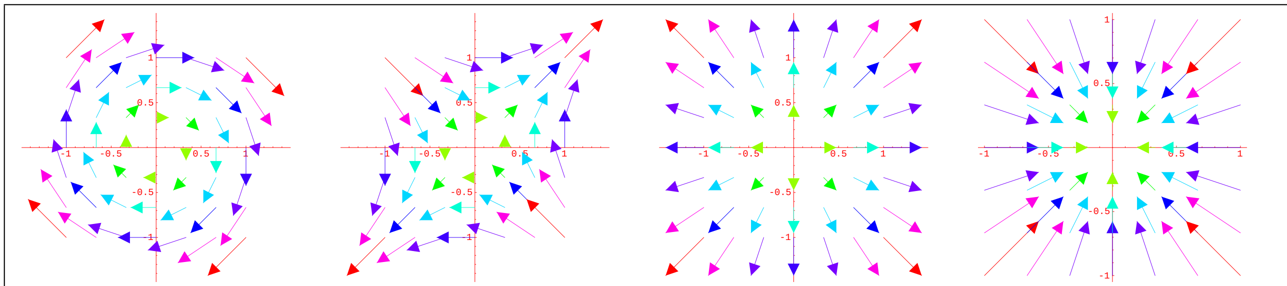
Example. Find a closed formula for the solution of the system

$$\begin{aligned} \dot{x}_1 &= x_1 + 2x_2 \\ \dot{x}_2 &= 4x_1 + 3x_2 \end{aligned}$$

with $\vec{x}(0) = \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The system can be written as $\dot{x} = Ax$ with $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. The matrix A has the eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to the eigenvalue -1 and the eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to the eigenvalue 5.

Because $A\vec{v}_1 = -\vec{v}_1$, we have $\vec{v}_1(t) = e^{-t}\vec{v}$. Because $A\vec{v}_2 = 5\vec{v}_1$, we have $\vec{v}_2(t) = e^{5t}\vec{v}$. The vector $\vec{v}$ can be written as a linear-combination of $\vec{v}_1$ and $\vec{v}_2$: $\vec{v} = \frac{1}{3}\vec{v}_2 + \frac{2}{3}\vec{v}_1$. Therefore, $\vec{x}(t) = \frac{1}{3}e^{5t}\vec{v}_2 + \frac{2}{3}e^{-t}\vec{v}_1$.

Phase portraits. For differential equations $\dot{x} = f(x)$ in two dimensions, one can **draw the vector field** $x \mapsto f(x)$. The solution curve $x(t)$ is tangent to the vector $f(x(t))$ everywhere. The phase portraits together with some solution curves reveal much about the system. Examples are



The closed form solution like $x(t) = e^{At}x(0)$ for $\dot{x} = Ax$ does not give us much insight what happens. One wants to understand the solution quantitatively. We want to understand questions like: What happens in the long term? Is the origin stable? Are there periodic solutions. Can one decompose the system into simpler subsystems? We will see that **diagonalisation** allows to **understand the system**. By decomposing it into one-dimensional linear systems, it can be analyzed separately. In general "understanding" can mean different things:

Plotting phase portraits.
Computing solutions numerically and estimate the error.
Finding special solutions.
Predicting the shape of some orbits.
Finding regions which are invariant.

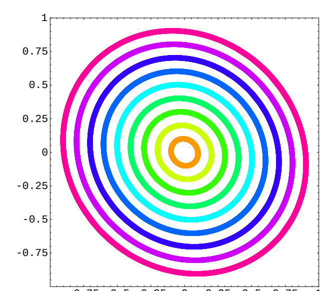
Finding special closed form solutions $x(t)$.
Finding a power series $x(t) = \sum_n a_n t^n$ in t.
Finding quantities which are unchanged along the flow (called "Integrals").
Finding quantities which increase along the flow (called "Lyapunov functions").

Linear stability: A linear dynamical system $\dot{x} = Ax$ with diagonalizable A is linearly stable if and only if $a_j = \mathrm{Re}(\lambda_j) < 0$ for all eigenvalues λ_j of A.

PROOF. We see that from the explicit solutions $y_j(t) = e^{a_j t}e^{ib_j t}y_j(0)$ in the basis consisting of eigenvectors. Now, $y(t) \to 0$ if and only if $a_j < 0$ for all j and $x(t) = Sy(t) \to 0$ if and only if $y(t) \to 0$.

What is the relation with discrete dynamical systems? From $\dot{x} = Ax$, we obtain $x(t+1) = Bx(t)$, with the matrix $B = e^A$. The eigenvalues of B are $\mu_j = e^{\lambda_j}$. Now $|\mu_j| < 1$ if and only if $\mathrm{Re}\lambda_j < 0$. The criterium for linear stability of discrete dynamical systems is compatible with the criterium for linear stability of $\dot{x} = Ax$.

Example: The system $\dot{x} = y, \dot{y} = -x$ can in vector form $v = (x, y)$ be written as $\dot{v} = Av$, with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The matrix A has the eigenvalues $i, -i$. After a coordinate transformation $w = S^{-1}v$ we get with $w = (a, b)$ the differential equations $\dot{a} = ia, \dot{b} = -ib$ which has the solutions $a(t) = e^{it}a(0), b(t) = e^{-it}b(0)$. The original coordates satisfy $x(t) = \cos(t)x(0) - \sin(t)y(0)$, $y(t) = \sin(t)x(0) + \cos(t)y(0)$.

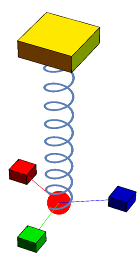


$\dot{x} = \lambda x$ for $\lambda = a + ib$ has the solution $x(t) = e^{at}e^{ibt}x(0)$ and length $|x(t)| = e^{at}|x(0)|$. For example, the differential equation $\dot{z} = iz$ has the solution e^{it} as well as the $\cos(t) + i\sin(t)$. Because the solutions of the differential equation are unique with $z(0) = 1$, we have an other verification of the **Euler formula** $e^{it} = \cos(t) + i\sin(t)$.

Derive that the harmonic oscillator $\ddot{x} = -c^2 x$ is solved by $x(t) = \cos(ct)x(0) + \sin(ct)\dot{x}(0)/c$..
Solution: $\dot{x} = y, \dot{y} = -c^2 x$ and in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

and because A has eigenvalues $\pm ic$, the new coordinates move as $a(t) = e^{ict}a(0)$ and $b(t) = e^{-ict}b(0)$. Writing this in the original coordinates $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$ and fixing the constants gives $x(t), y(t)$.



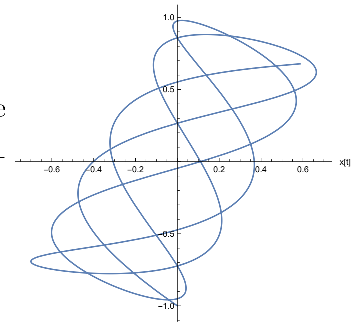
The **spinner** is a rigid body attached to a spring aligned around the z-axes. The body can rotate around the z-axes and bounce up and down. The two motions are coupled in the following way: when the spinner winds up in the same direction as the spring, the spring gets tightened and the body gets a lift. If the spinner winds up to the other direction, the spring becomes more relaxed and the body is lowered. Instead of reducing the system to a 4D first order system, system $\frac{d}{dt}\vec{x} = A\vec{x}$, we will keep the second time derivative and diagonalize the 2D system $\frac{d^2}{dt^2}\vec{x} = A\vec{x}$, where we know how to solve the one dimensional case $\frac{d^2}{dt^2}v = -\lambda v$ as $v(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)$ with constants A, B depending on the initial conditions, $v(0), \dot{v}(0)$.

x is the angle and y the height of the body. We put the coordinate system so that $y = 0$ is the point, where the body stays at rest if $x = 0$. We assume that if the spring is winded up with an angle x, this produces an upwards force x and a momentum force $-3x$. We furthermore assume that if the body is at position y, then this produces a momentum y onto the body and an upwards force y. The differential equations

$$\begin{aligned} \ddot{x} &= -3x + y \\ \ddot{y} &= -y + x \end{aligned} \qquad \text{can be written as } \ddot{v} = Av = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} v.$$

$w = S^{-1}v$ is obtained with getting the eigenvalues and eigenvectors of A: $\lambda_1 = -2 - \sqrt{2}$, $\lambda_2 = -2 + \sqrt{2}$
$v_1 = \begin{bmatrix} -1 - \sqrt{2} \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} -1 + \sqrt{2} \\ 1 \end{bmatrix}$ so that $S = \begin{bmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 & 1 \end{bmatrix}$.
We solve it $\ddot{a} = \lambda_1 a, \ddot{b} = \lambda_2 b$ in good coordinates $\begin{bmatrix} a \\ b \end{bmatrix} = S^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$.
$a(t) = A\cos(\omega_1 t) + B\sin(\omega_1)t$, $\omega_1 = \sqrt{-\lambda_1}$, $b(t) = C\cos(\omega_2 t) + D\sin(\omega_2)t$, $\omega_2 = \sqrt{-\lambda_2}$.

In the original coordinates, it is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$. At $t = 0$ we know $x(0), y(0), \dot{x}(0), \dot{y}(0)$. This fixes the constants in $x(t) = A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t)$
$+A_2\cos(\omega_2 t) + B_2\sin(\omega_2 t)$. The curve $(x(t), y(t))$ traces a Lyssajoux curve:



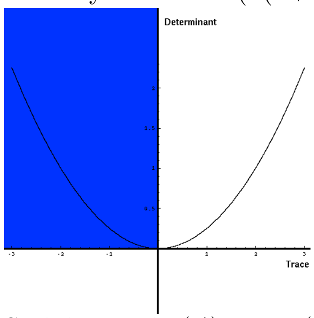
Asymptotic stability: $\dot{x} = Ax$ is asymptotically stable if and only if $\text{Re}(\lambda_i) < 0$ for all i.

A linear system $\dot{x} = Ax$ in the 2D plane is asymptotically stable if and only if $\det(A) > 0$ and $\text{tr}(A) < 0$.
PROOF. If both eigenvalues λ_1, λ_2 are real, then both being negative is equivalent to $\lambda_1\lambda_2 = \det(A) > 0$ and $\text{tr}(A) = \lambda_1 + \lambda_2 < 0$. If $\lambda_1 = a + ib, \lambda_2 = a - ib$, then a negative a is equivalent to $\lambda_1 + \lambda_2 = 2a < 0$ and $\lambda_1\lambda_2 = a^2 + b^2 > 0$.

Compare: Both trace and the determinant are independent of the basis and can be computed fast, and are real if A is real. It is therefore convenient to determine the region in the tr $-$ det-plane, where continuous or discrete dynamical systems are asymptotically stable. While the continuous dynamical system is related to a discrete system, it is important not to mix these two situations up.

Continuous dynamical system.

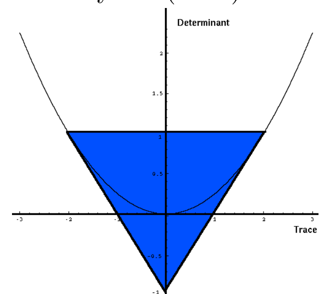
Stability of $\dot{x} = Ax$ $(x(t+1) = e^A x(t))$.



Stability in $\det(A) > 0, \mathrm{tr}(A) > 0$
Stability if $\mathrm{Re}(\lambda_1) < 0$, $\mathrm{Re}(\lambda_2) < 0$.

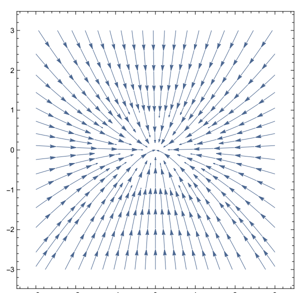
Discrete dynamical system.

Stability of $x(t+1) = Ax$

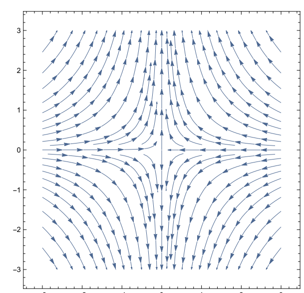


Stability in $|\mathrm{tr}(A)| - 1 < \det(A) < 1$
Stability if $|\lambda_1| < 1$,$|\lambda_2| < 1$.

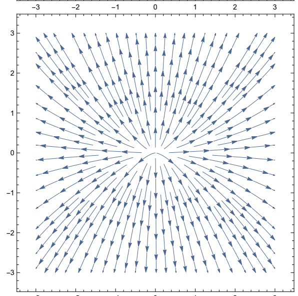
In two dimensions we can plot the vector field, draw some trajectories:



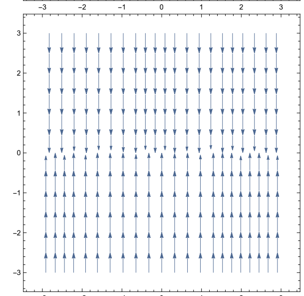
$\lambda_1 < 0$
$\lambda_2 < 0$,
i.e $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$



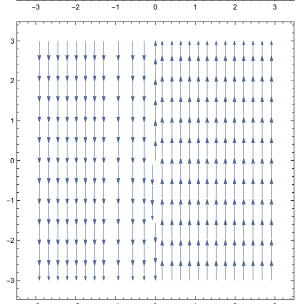
$\lambda_1 < 0$
$\lambda_2 > 0$,
i.e $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$



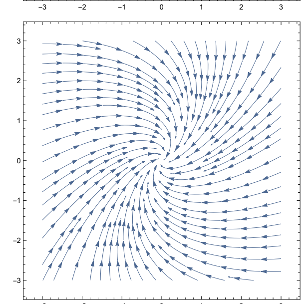
$\lambda_1 > 0$
$\lambda_2 > 0$,
i.e $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



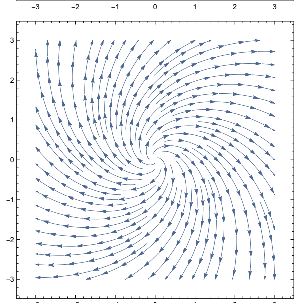
$\lambda_1 = 0$
$\lambda_2 < 0$,
i.e $A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$



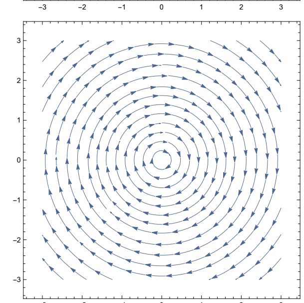
$\lambda_1 = 0$
$\lambda_2 = 0$,
i.e $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$



$\lambda_1 = a + ib, a < 0$
$\lambda_2 = a - ib$,
i.e $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 = a + ib, a > 0$
$\lambda_2 = a - ib$,
i.e $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 = ib$
$\lambda_2 = -ib$,
i.e $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

NONLINEAR DYNAMICAL SYSTEMS O.Knill, Math 21b

SUMMARY. For linear ordinary differential equations $\dot{x} = Ax$, the eigenvalues and eigenvectors of A determine the dynamics completely if A is diagonalizable. For nonlinear systems, explicit formulas for solutions are no more available in general. It can even happen that orbits go off to infinity in finite time like in the case of $\dot{x} = x^2$ which is solved by is $x(t) = -1/(t - x(0))$. With $x(0) = 1$, the solution "reaches infinity" at time $t = 1$. Linearity is often too crude. The exponential growth $\dot{x} = ax$ of a bacteria colony for example is slowed down due to the lack of food and the **logistic model** $\dot{x} = ax(1 - x/M)$ would be more accurate, where M is the population size for which bacteria starve so much that the growth has stopped: $x(t) = M$, then $\dot{x}(t) = 0$. Even so explicit solution formulas are no more available, nonlinear systems can still be investigated using linear algebra. In two dimensions $\dot{x} = f(x, y), \dot{y} = g(x, y)$, where "chaos" can not happen, the analysis of **equilibrium points** and **linear approximation** in general allows to understand the system quite well. This analysis also works to understand higher dimensional systems which can be "chaotic".

EQUILIBRIUM POINTS. A vector $\vec{x}_0$ is called an **equilibrium point** of $\frac{d}{dt}\vec{x} = f(\vec{x})$ if $f(\vec{x}_0) = 0$. If we start at an equilibrium point $x(0) = x_0$ then $x(t) = x_0$ for all times t. The Murray system $\dot{x} = x(6 - 2x - y), \dot{y} = y(4 - x - y)$ for example has the four equilibrium points $(0, 0), (3, 0), (0, 4), (2, 2)$.

JACOBIAN MATRIX. If x_0 is an equilibrium point for $\dot{x} = f(x)$ then $[A]_{ij} = \frac{\partial}{\partial x_j} f_i(x)$ is called the **Jacobian** at x_0. For two dimensional systems

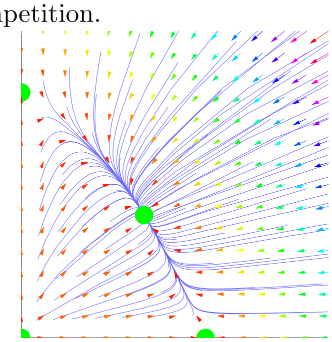
$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \qquad \text{this is the } 2 \times 2 \text{ matrix} \qquad A = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{bmatrix}.$$

The linear ODE $\dot{y} = Ay$ with $y = x - x_0$ approximates the nonlinear system well near the equilibrium point. The Jacobian is the linear approximation of $F = (f, g)$ near x_0.

VECTOR FIELD. In two dimensions, we can draw the vector field by hand: attaching a vector $(f(x, y), g(x, y))$ at each point (x, y). To find the equilibrium points, it helps to draw the **nullclines** $\{f(x, y) = 0\}, \{g(x, y) = 0\}$. The equilibrium points are located on intersections of nullclines. The eigenvalues of the Jacobeans at equilibrium points allow to draw the vector field near equilibrium points. This information is sometimes enough to draw the vector field **by hand**.

MURRAY SYSTEM. This system $\dot{x} = x(6 - 2x - y), \dot{y} = y(4 - x - y)$ has the nullclines $x = 0, y = 0, 2x + y = 6, x + y = 4$. There are 4 equilibrium points $(0, 0), (3, 0), (0, 4), (2, 2)$. The Jacobian matrix of the system at the point (x_0, y_0) is $\begin{bmatrix} 6 - 4x_0 - y_0 & -x_0 \\ -y_0 & 4 - x_0 - 2y_0 \end{bmatrix}$. Note that without interaction, the two systems would be logistic systems $\dot{x} = x(6 - 2x), \dot{y} = y(4 - y)$. The additional $-xy$ is the competition.

Equilibrium	Jacobean	Eigenvalues	Nature of equilibrium
(0,0)	$\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$	$\lambda_1 = 6, \lambda_2 = 4$	Unstable source
(3,0)	$\begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix}$	$\lambda_1 = -6, \lambda_2 = 1$	Hyperbolic saddle
(0,4)	$\begin{bmatrix} 2 & 0 \\ -4 & -4 \end{bmatrix}$	$\lambda_1 = 2, \lambda_2 = -4$	Hyperbolic saddle
(2,2)	$\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$	$\lambda_i = -3 \pm \sqrt{5}$	Stable sink



WITH MATHEMATICA Plotting the vector field:

```
Needs["VectorFieldPlots‘‘"]
f[x_,y_]:={x(6-2x-y),y(5-x-y)};VectorFieldPlot[f[x,y],{x,0,4},{y,0,4}]
```

Finding the equilibrium solutions:

```
Solve[{x(6-2x-y)==0,y(5-x-y)==0},{x,y}]
```

Finding the Jacobian and its eigenvalues at $(2, 2)$:

```
A[{x_,y_}]:={{6-4x,-x},{-y,5-x-2y}};Eigenvalues[A[{2,2}]]
```

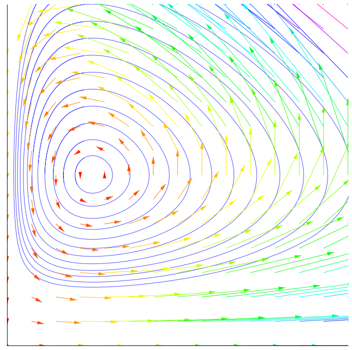
Plotting an orbit:

```
NDSolve[{x'[t]==x[t](6-2x[t]-y[t]),y'[t]==y[t](5-x[t]-y[t]),x[0]==1,y[0]==2},{x,y},{t,0,1}]
ParametricPlot[Evaluate[{x[t],y[t]}/.S],{t,0,1},AspectRatio->1,AxesLabel->{"x[t]","y[t]"}]
```

VOLTERRA-LODKA SYSTEMS are systems of the form

$$\begin{aligned}\dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy\end{aligned}$$

This example has equilibrium points $(0,0)$ and $(1/2,1)$.



It describes a predator-pray situation like for example a shrimp-shark population. The shrimp population $x(t)$ becomes smaller with more sharks. The shark population grows with more shrimp. Volterra explained so first the oscillation of fish populations in the Mediterrian sea.

EXAMPLE: HAMILTONIAN SYSTEMS are systems of the form

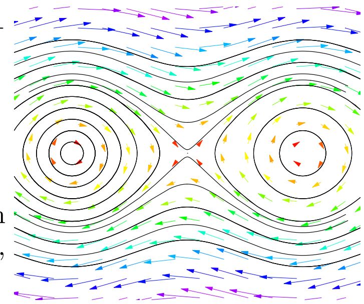
$$\begin{aligned}\dot{x} &= \partial_y H(x,y) \\ \dot{y} &= -\partial_x H(x,y)\end{aligned}$$

where H is called the **energy**. Usually, x is the position and y the momentum.

THE PENDULUM: $H(x,y) = y^2/2 - \cos(x)$.

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\sin(x)\end{aligned}$$

x is the angle between the pendulum and y-axes, y is the angular velocity, $\sin(x)$ is the potential.



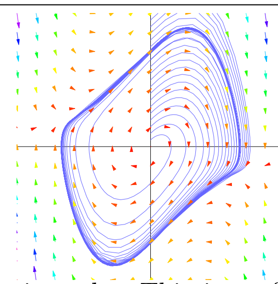
Hamiltonian systems preserve energy $H(x,y)$ because $\frac{d}{dt}H(x(t),y(t)) = \partial_x H(x,y)\dot{x} + \partial_y H(x,y)\dot{y} = \partial_x H(x,y)\partial_y H(x,y) - \partial_y H(x,y)\partial_x H(x,y) = 0$. Orbits stay on level curves of H.

EXAMPLE: LIENHARD SYSTEMS are differential equations of the form $\ddot{x} + \dot{x}F'(x) + G'(x) = 0$. With $y = \dot{x} + F(x), G'(x) = g(x)$, this gives

$$\begin{aligned}\dot{x} &= y - F(x) \\ \dot{y} &= -g(x)\end{aligned}$$

VAN DER POL EQUATION $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$ appears in electrical engineering, biology or biochemistry. Since $F(x) = x^3/3 - x, g(x) = x$.

$$\begin{aligned}\dot{x} &= y - (x^3/3 - x) \\ \dot{y} &= -x\end{aligned}$$

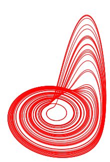


Lienhard systems have **limit cycles**. A trajectory always ends up on that limit cycle. This is useful for engineers, who need oscillators which are stable under changes of parameters. One knows: if $g(x) > 0$ for $x > 0$ and F has exactly three zeros $0, a, -a$, $F'(0) < 0$ and $F'(x) \geq 0$ for $x > a$ and $F(x) \to \infty$ for $x \to \infty$, then the corresponding Lienhard system has exactly one stable limit cycle.

CHAOS can occur for systems $\dot{x} = f(x)$ in three dimensions. For example, $\ddot{x} = f(x,t)$ can be written with $(x,y,z) = (x,\dot{x},t)$ as $(\dot{x},\dot{y},\dot{z}) = (y, f(x,z), 1)$. The system $\ddot{x} = f(x,\dot{x})$ becomes in the coordinates $(x,\dot{x})$ the ODE $\dot{x} = f(x)$ in four dimensions. The term **chaos** has no uniform definition, but usually means that one can find a copy of a random number generator embedded inside the system. Chaos theory is more than 100 years old. Basic insight had been obtained by Poincaré. During the last 30 years, the subject exploded to its own branch of physics, partly due to the availability of computers.

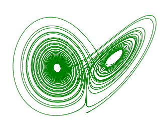
ROESSLER SYSTEM

$$\begin{aligned}\dot{x} &= -(y+z) \\ \dot{y} &= x + y/5 \\ \dot{z} &= 1/5 + xz - 5.7z\end{aligned}$$



LORENTZ SYSTEM

$$\begin{aligned}\dot{x} &= 10(y-x) \\ \dot{y} &= -xz + 28x - y \\ \dot{z} &= xy - \frac{8z}{3}\end{aligned}$$

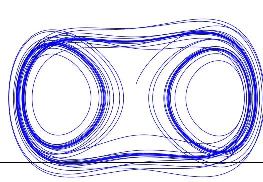


These two systems are examples, where one can observe **strange attractors**.

THE DUFFING SYSTEM

$\ddot{x} + \frac{\dot{x}}{10} - x + x^3 - 12\cos(t) = 0$

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -y/10 - x + x^3 - 12\cos(z) \\ \dot{z} &= 1\end{aligned}$$



The Duffing system models a metallic plate between magnets. Other chaotic examples can be obtained from mechanics like the **driven pendulum** $\ddot{x} + \sin(x) - \cos(t) = 0$.

A linear space X has the property that $\boxed{\text{if } f, g \text{ are in } X, \text{ then } f+g, \lambda f \text{ and a zero vector" } 0 \text{ are in } X.}$

- P_n, the space of all polynomials of degree n.
- The space P of all polynomials.
- C^∞, the space of all smooth functions.
- C^∞_{per} the space of all 2π periodic functions.

In all these function spaces, the function $f(x) = 0$ which is constant 0 is the zero function.

A map T on a linear space X is called **a linear transformation** if the following three properties hold $\boxed{T(x+y) = T(x) + T(y), T(\lambda x) = \lambda T(x) \text{ and } T(0) = 0.}$ Examples are:

- $Df(x)$=$f'(x)$ on C^∞
- $Tf(x) = \int_0^x f(x)\, dx$ on C^∞
- $Tf(x) = f(2x)$
- $Tf(x) = \sin(x)f(x)$
- $Tf(x) = \sin(x)f(x)$ on C^∞
- $Tf(x) = 5f(x)$
- $Tf(x) = f(x-1)$
- $Tf(x) = e^t \int_0^x e^{-t} f(t)\, dt$

Subspaces, eigenvalues, basis, kernel and image are defined as before

X linear subspace	$f, g \in X, f+g \in X, \lambda f \in X, 0 \in X$.
T linear transformation	$T(f+g) = T(f) + T(g), T(\lambda f) = \lambda T(f), T(0) = 0$.
$f_1, f_2, ..., f_n$ linear independent	$\sum_i c_i f_i = 0$ implies $f_i = 0$.
$f_1, f_2, ..., f_n$ span X	Every f is of the form $\sum_i c_i f_i$.
$f_1, f_2, ..., f_n$ basis of X	linear independent and span.
T has eigenvalue λ	$Tf = \lambda f$
kernel of T	$\{Tf = 0\}$
image of T	$\{Tf \mid f \in X\}$.

Some concepts do not work without modification. Example: $\det(T)$ or $\mathrm{tr}(T)$ are not always defined for linear transformations in infinite dimensions. The concept of a basis in infinite dimensions also needs to be defined properly.

The linear map $Df(x) = f'(x)$ can be iterated: $D^n f = f^{(n)}$ is the n'th derivative. It is a differential operator which allows to write differential equations like $f'' - f' = g$ in the same way than systems $A\vec{x} = \vec{b}$.

The kernel of D on C^∞ consists of all functions which satisfy $f'(x) = 0$. These are the constant functions. The kernel is one dimensional. The image is the entire space X because we can solve $Df = g$ by integration. You see that in infinite dimension, the fact that the image is equal to the codomain is not equivalent that the kernel is trivial.

Example: solve

$$Df = g\ .$$

The linear transformation T has a one dimensional kernel, the linear space of constant functions. The system $Df = g$ has therefore infinitely many solutions. Indeed, the solutions are of the form $f = G + c$, where F is the anti-derivative of g.

To solve

$$T(f) = (D - \lambda)f = g \; .$$

one can find f with the important formula

$$\boxed{f(x) = Ce^{\lambda x} + e^{\lambda x} \int_0^x g(x)e^{-\lambda x}\, dx}$$

as one can see by differentiating f: check $f' = \lambda f + g$. This is an important step because if we can invert T, we can invert also products $T_k T_{k-1} ... T_1$ and so solve $p(D)f = g$ for any polynomial p.

Find the eigenvectors to the eigenvalue λ of the operator D on $C^\infty(T)$. We have to solve

$$Df = \lambda f \; .$$

We see that $f(x) = e^\lambda(x)$ is a solution. But it is only a periodic solution if $\lambda = 2k\pi i$. Every number $\lambda = 2\pi k i$ is an eigenvalue. Eigenvalues are "quantized".

When we solved the harmonic oscillator differential equation

$$D^2 f + f = 0 \; .$$

last week, we actually saw that the transformation $T = D^2 + 1$ has a two dimensional kernel. It is spanned by the functions $f_1(x) = \cos(x)$ and $f_2(x) = \sin(x)$. Every solution to the differential equation is of the form $c_1 \cos(x) + c_2 \sin(x)$.

Problem 1) Which of the following maps are linear operators?
a) $T(f)(x) = x^2 f(x - 4)$
b) $T(f)(x) = f'(x)^2$
c) $T = D^2 + D + 1$ meaning $T(f)(x) = f''(x) + f'(x) + f(x)$.
d) $T(f)(x) = e^x \int_0^x e^{-t} f(t)\, dt$.

Problem 2) a) What is the kernel and image of the linear operators $T = D + 3$ and $D - 2$? Use this to find the kernel of $p(D)$ for $p(x) = x^2 + x - 6$?
b) Verify that $f(x) = xe^{-x^2/2}$ is in the kernel of the differential operator $T = D + x$.

Problem 3) In quantum mechanics, the operator $P = iD$ is called the **momentum operator** and the operator $Qf(x) = xf(x)$ is the **position operator**.
a) Verify that every λ is an eigenvalue of P. What is the eigenfunction?
b) What operator is $[Q, P] = QP - PQ$?

Problem 4) The differential equation $f' - 3f = \sin(x)$ can be written as

$$Tf = g$$

with $T = D - 3$ and $g = \sin$. We need to invert the operator T. Verify that

$$Hg = e^{3x} \int_0^x e^{-3t} g(t)\, dt$$

is an inverse of T. In other words, show that the function $f = Hg$ satisfies $Tf = g$.

Problem 5) The operator

$$Tf(x) = -f''(x) + x^2 f(x)$$

is called the **energy operator** of the **quantum harmonic oscillator**.
a) Check that $f(x) = e^{-x^2/2}$ is an eigenfunction of T. What is the eigenvalue?
b) Verify that $f(x) = xe^{-x^2/2}$ is an eigenfunction of T. What is the eigenvalue?

$$x' - \lambda x = 0 \qquad x(t) = Ce^{\lambda t}$$

This first order ODE is by far the most important differential equation. A linear system of differential equation $x'(t) = Ax(t)$ reduces to this after diagonalization. We can rewrite the differential equation $x' - \lambda x = 0$ as $(D - \lambda)x = 0$. That is x is in the kernel of $D - \lambda$. An other interpretation is that $\exp(\lambda x)$ is an eigenfunction of D belonging to the eigenvalue λ. This differential equation describes exponential growth or exponential decay.

$$x'' + k^2 x = 0 \qquad x(t) = A\cos(kt) + B\sin(kt)$$

This second order ODE is by far the second most important differential equation. Any linear system of differential equations $x''(t) = Ax(t)$ reduces to this by diagonalization. We can rewrite the differential equation $x'' + k^2x = 0$ as $(D^2 + k^2)x = 0$. That is, x is in the kernel of $D^2 + k^2$. An other interpretation is that x is an **eigenfunction** of D^2 belonging to the eigenvalue $-k^2$. This differential equation describes oscillations or waves.

OPERATOR METHOD. A general method to find solutions to $p(D)f = g$ is to factor the polynomial $p(D) = (D - \lambda_1) \cdots (D - \lambda_n)x = g$, then invert each factor to get

$$x = (D - \lambda_n)^{-1} \ldots\ldots (D - \lambda_1)^{-1} g$$

where

$$(D - \lambda)^{-1} g = Ce^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda s} g(s)\, ds$$

COOKBOOK METHOD. The operator method always works. But it can produce a considerable amount of work. Engineers therefore rely also on cookbook recipes. The solution of an inhomogeneous differential equation $p(D)x = g$ is found by first finding the **homogeneous solution** x_h which is the solution to $p(D)x = 0$. Then a **particular solution** x_p of the system $p(D)x = g$ is found by an educated guess. This method is often much faster but it requires to know the "recipes". Fortunately, it is quite easy: as a rule of thumb: feed in the same class of functions which you see on the right hand side and if the right hand side should contain a function in the kernel of $p(D)$, try with a function multiplied by t. The general solution of the system $p(D)x = g$ is $x = x_h + x_p$.

FINDING THE HOMOGENEOUS SOLUTION. $p(D) = (D - \lambda_1)(D - \lambda_2) = D^2 + bD + c$. The next table covers all cases for homogeneous second order differential equations $x'' + px' + q = 0$.

$\lambda_1 \neq \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
$\lambda_1 = \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$
$ik = \lambda_1 = -\lambda_2$ imaginary	$C_1 \cos(kt) + C_2 \sin(kt)$
$\lambda_1 = a + ik, \lambda_2 = a - ik$	$C_1 e^{at} \cos(kt) + C_2 e^{at} \sin(kt)$

FINDING AN INHOMOGENEOUS SOLUTION. Inhomogeneous solutions can be found by applying the operator inversions with $C = 0$ or by an educated guess. For $x'' = g(t)$ we just integrate twice, otherwise, check with the following table:

$g(t) = a$ constant	$x(t) = A$ constant
$g(t) = at + b$	$x(t) = At + B$
$g(t) = at^2 + bt + c$	$x(t) = At^2 + Bt + C$
$g(t) = a\cos(bt)$	$x(t) = A\cos(bt) + B\sin(bt)$
$g(t) = a\sin(bt)$	$x(t) = A\cos(bt) + B\sin(bt)$
$g(t) = a\cos(bt)$ with $p(D)g = 0$	$x(t) = At\cos(bt) + Bt\sin(bt)$
$g(t) = a\sin(bt)$ with $p(D)g = 0$	$x(t) = At\cos(bt) + Bt\sin(bt)$
$g(t) = ae^{bt}$	$x(t) = Ae^{bt}$
$g(t) = ae^{bt}$ with $p(D)g = 0$	$x(t) = Ate^{bt}$
$g(t) = q(t)$ polynomial	$x(t) =$ polynomial of same degree

EXAMPLE 1: $$f'' = \cos(5x)$$

This is of the form $D^2 f = g$ and can be solved by inverting D which is integration: integrate a first time to get $Df = C_1 + \sin(5x)/5$. Integrate a second time to get
$$f = C_2 + C_1 t - \cos(5t)/25$$
This is the operator method in the case $\lambda = 0$.

EXAMPLE 2: $$f' - 2f = 2t^2 - 1$$

This homogeneous differential equation $f' - 5f = 0$ is hardwired to our brain. We know its solution is Ce^{2t}. To get a homogeneous solution, try $f(t) = At^2 + Bt + C$. We have to compare coefficients of $f' - 2f = -2At^2 + (2A - 2B)t + B - 2C = 2t^2 - 1$. We see that $A = -1, B = -1, C = 0$. The special solution is $-t^2 - t$. The complete solution is
$$f = -t^2 - t + Ce^{2t}$$

EXAMPLE 3: $$f' - 2f = e^{2t}$$

In this case, the right hand side is in the kernel of the operator $T = D - 2$ in equation $T(f) = g$. The homogeneous solution is the same as in example 2, to find the inhomogeneous solution, try $f(t) = Ate^{2t}$. We get $f' - 2f = Ae^{2t}$ so that $A = 1$. The complete solution is
$$f = te^{2t} + Ce^{2t}$$

EXAMPLE 4: $$f'' - 4f = e^t$$

To find the solution of the homogeneous equation $(D^2 - 4)f = 0$, we factor $(D-2)(D+2)f = 0$ and add solutions of $(D-2)f = 0$ and $(D+2)f = 0$ which gives $C_1 e^{2t} + C_2 e^{-2t}$. To get a special solution, we try Ae^t and get from $f'' - 4f = e^t$ that $A = -1/3$. The complete solution is
$$f = -e^t/3 + C_1 e^{2t} + C_2 e^{-2t}$$

EXAMPLE 5: $$f'' - 4f = e^{2t}$$

The homogeneous solution $C_1 e^{2t} + C_2 e^{-2t}$ is the same as before. To get a special solution, we can not use Ae^{2t} because it is in the kernel of $D^2 - 4$. We try Ate^{2t}, compare coefficients and get
$$f = te^{2t}/4 + C_1 e^{2t} + C_2 e^{-2t}$$

EXAMPLE 6: $$f'' + 4f = e^t$$

The homogeneous equation is a harmonic oscillator with solution $C_1 \cos(2t) + C_2 \sin(2t)$. To get a special solution, we try Ae^t compare coefficients and get

$$f = e^t/5 + C_1 \cos(2t) + C_2 \sin(2t)$$

EXAMPLE 7: $$f'' + 4f = \sin(t)$$

The homogeneous solution $C_1 \cos(2t) + C_2 \sin(2t)$ is the same as in the last example. To get a special solution, we try $A \sin(t) + B \cos(t)$ compare coefficients (because we have only even derivatives, we can even try $A \sin(t)$) and get $$f = \sin(t)/3 + C_1 \cos(2t) + C_2 \sin(2t)$$

EXAMPLE 8: $$f'' + 4f = \sin(2t)$$

The solution $C_1 \cos(2t) + C_2 \sin(2t)$ is the same as in the last example. To get a special solution, we can not try $A \sin(t)$ because it is in the kernel of the operator. We try $At \sin(2t) + Bt \cos(2t)$ instead and compare coefficients $f = t \sin(2t)/16 - t \cos(2t)/4 + C_1 \cos(2t) + C_2 \sin(2t)$

EXAMPLE 9: $$f'' + 8f' + 16f = \sin(5t)$$

The homogeneous solution is $C_1 e^{-4t} + C_2 te^{-4t}$. To get a special solution, we try $A \sin(5t) + B \cos(5t)$ compare coefficients and get $f = -40 \cos(5t)/41^2 + -9 \sin(t)/41^2 + C_1 e^{-4t} + C_2 te^{-4t}$

EXAMPLE 10: $$f'' + 8f' + 16f = e^{-4t}$$

The homogeneous solution is still $C_1 e^{-4t} + C_2 te^{-4t}$. To get a special solution, we can not try e^{-4t} nor te^{-4t} because both are in the kernel. Add an other t and try with $At^2 e^{-4t}$. $f = t^2 e^{-4t}/2 + C_1 e^{-4t} + C_2 te^{-4t}$

EXAMPLE 11: $$f'' + f' + f = e^{-4t}$$

By factoring $D^2 + D + 1 = (D - (1 + \sqrt{3}i)/2)(D - (1 - \sqrt{3}i)/2)$ we get the homogeneous solution $C_1 e^{-t/2} \cos(\sqrt{3}t/2) + C_2 e^{-t/2} \sin(\sqrt{3}t/2)$. For a special solution, try Ae^{-4t}. Comparing coefficients gives $A = 1/13$. $f = e^{-4t}/13 + C_1 e^{-t/2} \cos(\sqrt{3}t/2) + C_2 e^{-t/2} \sin(\sqrt{3}t/2)$

O. Knill, April 7, 2007-2018

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS. $Df = Tf = f'$ is a linear map on the space of smooth functions C^∞. If $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is a polynomial, then $p(D) = a_0 + a_1 D + \cdots + a_n D^n$ is a linear map on $C^\infty(R)$ too. We will see here how to find the general solution of $p(D)f = g$.
EXAMPLE. For $p(x) = x^2 - x + 6$ and $g(x) = \cos(x)$ the problem $p(D)f = g$ is the differential equation $f''(x) - f'(x) - 6f(x) = \cos(x)$. It has the solution $c_1 e^{-2x} + c_2 e^{3x} - (\sin(x) + 7\cos(x))/50$, where c_1, c_2 are arbitrary constants. How can one find these solutions?

THE IDEA. In general, a differential equation $p(D)f = g$ has many solution. For example, for $p(D) = D^3$, the equation $D^3 f = 0$ has solutions $(c_0 + c_1 x + c_2 x^2)$. The constants come because we integrated three times. Integrating means applying D^{-1} but because D has as the kernel the constant functions, integration gives a one dimensional space of anti-derivatives. (We can add a constant to the result and still have an anti-derivative). In order to solve $D^3 f = g$, we integrate g three times. One can generalize this idea by writing $T = p(D)$ as a product of simpler transformations which we can invert. These simpler transformations have the form $(D - \lambda)f = g$.

FINDING THE KERNEL OF A POLYNOMIAL IN D. How do we find a basis for the kernel of $Tf = f'' + 2f' + f$? The linear map T can be written as a polynomial in D which means $T = D^2 - D - 2 = (D + 1)(D - 2)$. The kernel of T contains the kernel of $D - 2$ which is one-dimensional and spanned by $f_1 = e^{2x}$. The kernel of $T = (D - 2)(D + 1)$ also contains the kernel of $D + 1$ which is spanned by $f_2 = e^{-x}$. The kernel of T is therefore two dimensional and spanned by e^{2x} and e^{-x}.

THEOREM: If $T = p(D) = D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0$ on C^∞ then $\dim(\ker(T)) = n$.

PROOF. $T = p(D) = \prod(D - \lambda_j)$, where λ_j are the roots of the polynomial p. The kernel of T contains the kernel of $D - \lambda_j$ which is spanned by $f_j(t) = e^{\lambda_j t}$. In the case when we have a factor $(D - \lambda_j)^k$ of T, then we have to consider the kernel of $(D - \lambda_j)^k$ which is $q(t)e^{\lambda t}$, where q is a polynomial of degree $k - 1$. For example, the kernel of $(D - 1)^3$ consists of all functions $(a + bt + ct^2)e^t$.

SECOND PROOF. Write this as $A\dot{g} = 0$, where A is a $n \times n$ matrix and $g = \left[f, \dot{f}, \cdots, f^{(n-1)}\right]^T$, where $f^{(k)} = D^k f$ is the k'th derivative. The linear map $T = AD$ acts on vectors of functions. If all eigenvalues λ_j of A are different (they are the same λ_j as before), then A can be diagonalized. Solving the diagonal case $BD = 0$ is easy. It has a n dimensional kernel of vectors $F = [f_1, \ldots, f_n]^T$, where $f_i(t) = t$. If $B = SAS^{-1}$, and F is in the kernel of BD, then SF is in the kernel of AD.

REMARK. The result can be generalized to the case, when a_j are functions of x. Especially, $Tf = g$ has a solution, when T is of the above form. It is important that the function in front of the highest power D^n is bounded away from 0 for all t. For example $xDf(x) = e^x$ has no solution in C^∞, because we can not integrate e^x/x. An example of a ODE with variable coefficients is the **Sturm-Liouville** eigenvalue problem $T(f)(x) = a(x)f''(x) + a'(x)f'(x) + q(x)f(x) = \lambda f(x)$ like for example the Legendre differential equation $(1 - x^2)f''(x) - 2xf'(x) + n(n + 1)f(x) = 0$.

BACKUP

- Equations $Tf = 0$, where $T = p(D)$ form **linear differential equations with constant coefficients** for which we want to understand the solution space. Such equations are called **homogeneous**. **Solving the equation includes finding a basis of the kernel of T.** In the above example, a general solution of $f'' + 2f' + f = 0$ can be written as $f(t) = a_1 f_1(t) + a_2 f_2(t)$. If we fix two values like $f(0), f'(0)$ or $f(0), f(1)$, the solution is unique.

- If we want to solve $Tf = g$, an **inhomogeneous equation** then T^{-1} is not unique because we have a kernel. If g is in the image of T there is at least one solution f. The general solution is then $f + \ker(T)$. For example, for $T = D^2$, which has C^∞ as its image, we can find a solution to $D^2 f = t^3$ by integrating twice: $f(t) = t^5/20$. The kernel of T consists of all linear functions $at + b$. The general solution to $D^2 = t^3$ is $at + b + t^5/20$. The integration constants parameterize actually the kernel of a linear map.

THE SYSTEM $Tf = (D-\lambda)f = g$ has the general solution $\boxed{ce^{\lambda x} + e^{\lambda x}\int_0^x e^{-\lambda t} g(t)\, dt}$.

THE SOLUTION OF $(D-\lambda)^k f = g$ is obtained by applying $(D-\lambda)^{-1}$ several times on g. In particular, for $g = 0$, we get $\boxed{\text{the kernel of } (D-\lambda)^k \text{ as } (c_0 + c_1 x + ... + c_{k-1}x^{k-1})e^{\lambda x}}$.

$\boxed{\text{THEOREM. The inhomogeneous } p(D)f = g \text{ has an } n\text{-dimensional space of solutions in } C^\infty(\mathbf{R}).}$

PROOF. To solve $Tf = p(D)f = g$, we write the equation as $(D-\lambda_1)^{k_1}(D-\lambda_2)^{k_2}\cdots(D-\lambda_n)^{k_n} f = g$. Since we know how to invert each $T_j = (D-\lambda_j)^{k_j}$, we can construct the general solution by inverting one factor T_j of T one after another.

Often we can find directly a special solution f_1 of $p(D)f = g$ and get the general solution as $f_1 + f_h$, where f_h is in the n-dimensional kernel of T.

EXAMPLE 1) $Tf = e^{3x}$, where $T = D^2 - D = D(D-1)$. We first solve $(D-1)f = e^{3x}$. It has the solution $f_1 = ce^x + e^x\int_0^x e^{-t}e^{3t}\, dt = c_2 e^x + e^{3x}/2$. Now solve $Df = f_1$. It has the solution $\boxed{c_1 + c_2 e^x + e^{3x}/6}$.

EXAMPLE 2) $Tf = \sin(x)$ with $T = (D^2 - 2D + 1) = (D-1)^2$. We see that $\cos(x)/2$ is a special solution. The kernel of $T = (D-1)^2$ is spanned by xe^x and e^x so that the general solution is $\boxed{(c_1 + c_2 x)e^x + \cos(x)/2}$.

EXAMPLE 3) $Tf = x$ with $T = D^2+1 = (D-i)(D+i)$ has the special solution $f(x) = x$. The kernel is spanned by e^{ix} and e^{-ix} or also by $\cos(x), \sin(x)$. The general solution can be written as $\boxed{c_1\cos(x) + c_2\sin(x) + x}$.

EXAMPLE 4) $Tf = x$ with $T = D^4 + 2D^2 + 1 = (D-i)^2(D+i)^2$ has the special solution $f(x) = x$. The kernel is spanned by $e^{ix}, xe^{ix}, e^{-ix}, x^{-ix}$ or also by $\cos(x), \sin(x), x\cos(x), x\sin(x)$. The general solution can be written as $\boxed{(c_0 + c_1 x)\cos(x) + (d_0 + d_1 x)\sin(x) + x}$.

THESE EXAMPLES FORM 4 TYPICAL CASES.

CASE 1) $p(D) = (D-\lambda_1)...(D-\lambda_n)$ with real λ_i. The general solution of $p(D)f = g$ is the sum of a special solution and $\boxed{c_1 e^{\lambda_1 x} + ... + c_n e^{\lambda_n x}}$

CASE 2) $p(D) = (D-\lambda)^k$. The general solution is the sum of a special solution and a term $\boxed{(c_0 + c_1 x + ... + c_{k-1}x^{k-1})e^{\lambda x}}$

CASE 3) $p(D) = (D-\lambda)(D-\overline{\lambda})$ with $\lambda = a + ib$. The general solution is a sum of a special solution and a term $\boxed{c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx)}$

CASE 4) $p(D) = (D-\lambda)^k(D-\overline{\lambda})^k$ with $\lambda = a + ib$. The general solution is a sum of a special solution and $\boxed{(c_0 + c_1 x + ... + c_{k-1}x^{k-1})e^{ax}\cos(bx) + (d_0 + d_1 x + ... + d_{k-1}x^{k-1})e^{ax}\sin(bx)}$

We know this also from the eigenvalue problem for a matrix. We either have distinct real eigenvalues, or we have some eigenvalues with multiplicity, or we have pairs of complex conjugate eigenvalues which are distinct, or we have pairs of complex conjugate eigenvalues with some multiplicity.

CAS SOLUTION OF ODE's: Example: DSolve[f''[x] − f'[x] == Exp[3x], f[x], x]

INFORMAL REMARK. Operator methods can also be useful for ODEs with variable coefficients. For example, $T = H - 1 = D^2 - x^2 - 1$, the **quantum harmonic oscillator**, can be written as $T = A^*A = AA^* + 2$ with a **creation operator** $A^* = (D - x)$ and **annihilation operator** $A = (D + x)$. To see this, use the **commutation relation** $Dx - xD = 1$. The kernel $f_0 = Ce^{-x^2/2}$ of $A = (D + x)$ is also the kernel of T and so an eigenvector of T and H. It is called the **vacuum**. $\boxed{\text{If } f \text{ is an eigenvector of } H \text{ with } Hf = \lambda f, \text{ then } A^*f \text{ is an eigenvector with eigenvalue } \lambda + 2}$. Proof. Because $HA^* - A^*H = [H, A^*] = 2A^*$, we have $H(A^*f) = A^*Hf + [H, A^*]f = A^*\lambda f + 2A^*f = (\lambda + 2)(A^*f)$. We obtain all eigenvectors $f_n = A^*f_{n-1}$ of eigenvalue $1 + 2n$ by applying iteratively the creation operator A^* on the vacuum f_0. Because every function f with $\int f^2\, dx < \infty$ can be written uniquely as $f = \sum_{n=0}^\infty a_n f_n$, we can **diagonalize** H and solve $Hf = g$ with $f = \sum_n b_n/(1 + 2n) f_n$, where $g = \sum_n b_n f_n$.

INNER PRODUCT	Math 21b, O. Knill

DOT PRODUCT. With the **dot product** in $\mathbf{R}^n$, we were able to define **angles**, **length**, compute projections onto planes or reflections on lines. Especially recall that if $\vec{w}_1, ..., \vec{w}_n$ was an orthonormal set, then $\vec{v} = a_1\vec{w}_1 + ... + a_n\vec{w}_n$ with $a_i = \vec{v} \cdot \vec{w}_i$. This was the formula for the orthonormal projection in the case of an orthogonal set. We will aim to do the same for functions. But first we need to define a "dot product" for functions.

THE INNER PRODUCT. For piecewise smooth functions f, g on $[-\pi, \pi]$, we define the **inner product**

$$\boxed{\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx}$$

It plays the role of the dot product in $\mathbf{R}^n$. It has the same properties as the familiar dot product:

(i) $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$.
(ii) $||f||^2 = \langle f, g \rangle \geq 0$
(iii) $||f||^2 = 0$ if and only if f is identically 0

EXAMPLES.

- $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^{3/2} \, dx = \frac{1}{\pi} x^{5/2} \frac{2}{5} |_{-\pi}^{\pi} = \frac{4}{5}\sqrt{\pi^3}$.
- $f(x) = \sin^2(x)$, $g(x) = x^3$. Then $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(x) x^3 \, dx = ...?$

Before integrating, Ii is always a good idea to look for some symmetry. Can you see the result without doing the integral?

PROPERTIES. The

- **triangle inequality** $||f + g|| \leq ||f|| + ||g|$.
- the **Cauchy-Schwartz inequality** $|\langle f, g \rangle \leq ||f|| \, ||g||$
- as well as **Pythagoras theorem** $||f + g||^2 = ||f||^2 + ||g||^2$ for orthogonal functions

hold in the same way as they did in R^n. The proofs are identical.

ANGLE, LENGTH, DISTANCE, ORTHOGONALITY. With an inner product, we can do things as with the dot product:

- Compute the **angle** α between two functions f and g $\boxed{\cos(\alpha) = \frac{\langle f, g \rangle}{||f|| \cdot ||g||}}$
- Determine the **length** $\boxed{||f||^2 = \langle f, f \rangle}$
- Find and **distance** $\boxed{||f - g||}$ between two functions
- Project a function f onto a space of functions. $P(f) = < f, g_1 > g_1 + < f, g_2) g_2 ... + < f, g_n > g_n$ if the functions g_i are orthonormal.

Note that $||f|| = 0$ implies that f is identically 0. Two functions whose distance is zero are identical.

EXAMPLE: ANGLE COMPUTATION.

Problem: Find the angle between the functions $f(t) = t^3$ and $g(t) = t^4$.

Answer: The angle is 90°. This can be seen by symmetry. The integral on $[-\pi, 0]$ is the negative then the integral on $[0, \pi]$.

EXAMPLE: GRAM SCHMIDT ORTHOGONALIZATION.

Problem: Given a two dimensional plane spanned by $f_1(t) = 1, f_2(t) = t^2$, use Gram-Schmidt orthonormalization to get an orthonormal set.

Solution. The function $g_1(t) = 1/\sqrt{2}$ has length 1. To get an orthonormal function $g_2(t)$, we use the formula of the Gram-Schmidt orthogonalization process: first form

$$h_2(t) = f_2(t) - \langle f_2(t), g_1(t)\rangle g_1(t)$$

then get $g_2(t) = h_2(t)/||h_2(t)||$.

EXAMPLE: PROJECTION.

Problem: Project the function $f(t) = t$ onto the plane spanned by the functions $\sin(t), \cos(t)$.

EXAMPLE: REFLECTION.

Problem: Reflect the function $f(t) = \cos(t)$ at the line spanned by the function $g(t) = t$.
Solution: Let $c = ||g||$. The projection of f onto g is $h = \langle f, g\rangle g/c^2$. The reflection is $f + 2(h - f)$ as with vectors.

EXAMPLE: Verify that if $f(t)$ is a 2π periodic function, then f and its derivative f' are orthogonal.

Solution. Define $g(x,t) = f(x+t)$ and consider its length $l(t) = ||g(x,t)||$ when fixing t. The length does not change. So, differentiating $0 = l'(t) = d/dt\langle f(x+t), f(x+t)\rangle = \langle f'(x+t), f(x+t)\rangle + \langle f(x+t), f'(x+t)\rangle = 2\langle f'(x+t), f(x+t)\rangle$.

PROBLEMS.

1. Find the angle between $f(x) = \cos(x)$ and $g(x) = x^2$. (Like in $\mathbb{R}^n$, we define the angle between f and g to be $\arccos \frac{\langle f,g\rangle}{\|f\|\|g\|}$ where $\|f\| = \sqrt{\langle f, f\rangle}$.)

   Remarks. Use integration by parts twice to compute the integral. This is a good exercise if you feel a bit rusty about integration techniques. Feel free to double check your computation with the computer but try to do the computation by hand.

2. A function on $[-\pi, \pi]$ is called **even** if $f(-x) = f(x)$ for all x and **odd** if $f(-x) = -f(x)$ for all x. For example, $f(x) = \cos x$ is even and $f(x) = \sin x$ is odd.

   a) Verify that if f, g are even functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g\rangle = \frac{2}{\pi}\int_0^\pi f(x)g(x)\,dx$.

   b) Verify that if f, g are odd functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g\rangle = \frac{2}{\pi}\int_0^\pi f(x)g(x)\,dx$.

   c) Verify that if f is an even function on $[-\pi, \pi]$ and g is an odd function on $[-\pi, \pi]$, then $\langle f, g\rangle = 0$.

3. Which of the two functions $f(x) = \cos(x)$ or $g(x) = \sin(x)$ is closer to the function $h(x) = x^2$?

4. Determine the projection of the function $f(x) = x^2$ onto the "plane" spanned by the two orthonormal functions $g(x) = \cos(x)$ and $h(x) = \sin(x)$.

   Hint. You have computed the inner product between f and g already in problem 1). Think before you compute the inner product between f and h. There is no calculation necessary to compute $\langle f, h\rangle$.

5. Recall that $\cos(x)$ and $\sin(x)$ are orthonormal. Find the length of $f(x) = a\cos(x) + b\sin(x)$ in terms of a and b.

INTEGRATION TIPS FOR FINDING FOURIER SERIES Math 21b, O. Knill

USEFUL TRIGONOMETRIC FORMULAS:

$$\begin{aligned}
2\cos(nx)\cos(my) &= \cos(nx-my)+\cos(nx+my) \\
2\sin(nx)\sin(my) &= \cos(nx-my)-\cos(nx+my) \\
2\sin(nx)\cos(my) &= \sin(nx+my)+\sin(nx-my)
\end{aligned}$$

THE FOURIER SERIES OF $\cos^2(t)$ and $\sin^2(t)$.

$$\cos(2t) = \cos^2(t)-\sin^2(t) = 2\cos^2(t)-1 = 1-\sin^2(t)$$

Leads to the formulas

$$\cos^2(t) = (1+\cos(2t)/2$$

$$\sin^2(t) = (1-\cos(2t)/2$$

Note that these are the Fourier series of the function $f(t) = \cos^2(t)$ and $g(t) = \sin^2(t)$!

SYMMETRY.

- If you integrate an odd function over $[-\pi, \pi]$ you get 0.
- The product between an odd and an even function is an odd function.

INTEGRATION BY PART. Integrating the differentiation rule $(uv)' = u'v + vu'$ gives the partial integration formula:

$$\int uv'\ dt = uv - \int u'v\ dt$$

Examples:

$$\int t\sin(t)\ dt = -t\cos(t) + \int \cos(t)\ dt = \sin(t) - t\cos(t)\ .$$

$$\int t\cos(t)\ dt = t\sin(t) - \int \sin(t)\ dt = \cos(t) + t\sin(t)\ .$$

Sometimes you have repeat doing integration by part. For example, to derive the formulas

$$\int t^2\sin(t)\ dt = 2t\sin[t] - (t^2-2)\cos[t]\ .$$

$$\int t^2\cos(t)\ dt = 2t\cos[t] + (t^2-2)\sin[t]\ .$$

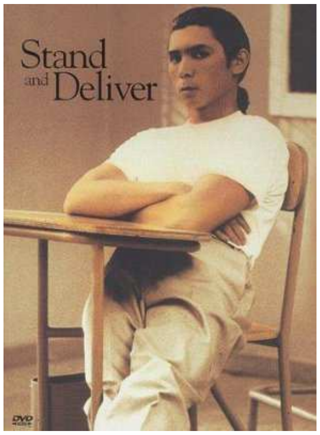
one has to integrate by part twice.

THE LENGTH OF THE FOURIER BASIS VECTORS. A frequently occurring definite integral:

$$\int_{-\pi}^{\pi} \cos^2(nt)\ dt = \pi$$

$$\int_{-\pi}^{\pi} \sin^2(nt)\ dt = \pi$$

These formulas can be derived also by noting that the two integrals must be the same because $\cos(nt) = \sin(nt + \pi/2)$. If one sums those two integrals, using $\cos^2(nt) + \sin^2(nt) = 1$ one gets 2π. So each integral must be π.



STAND AND DELIVER.

In the movie "Stand and Deliver" the following "Tic-Tac-Toe" method is used to compute the antiderivative of $f(x) = x^2 \sin(x)$.

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The result is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x)\ .$$

Find the Fourier series of the function $f(x) = x^2$. To do so, we use the Tic-Tac-Toe method to compute the integral $\int x^2 \cos(nx)\, dx$:

x^2	$\cos(nx)$	
$2x$	$\sin(nx)/n$	$\oplus$
2	$-\cos(nx)/n^2$	$\ominus$
0	$-\sin(nx)/n^3$	$\oplus$

The result is

$$x^2\sin(nx)/n + 2x\cos(nx)/n^2 - 2\sin(nx)/n^3$$

Cool.

Reminder: Piecewise smooth functions $f(x)$ on $[-\pi,\pi]$ form a linear space$X$. With an inner product in X

$$\boxed{\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx}$$

we can define angles, length and projections in X as we did before. Building a Fourier series is the task to write a function as a linear combination of $\sin(nx),\cos(nx)$ and a constant term. You have verified that the set of functions $\boxed{\cos(nx),\sin(nx),1/\sqrt{2}}$ form an orthonormal basis in X.

The Fourier coefficients of f are $a_0 = \langle f,1/\sqrt{2}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)/\sqrt{2}\,dx$, $a_n = \langle f,\cos(nt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$, $b_n = \langle f,\sin(nt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$.

The Fourier series is then $f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx)$

If f is odd: $f(x) = -f(-x)$ then f has a sin-series.
If f is even: $f(x) = -f(-x)$ then f has a cos-series.

The reason is that if you take the dot product between an odd and an even function, you integrate an odd function on the interval $[-\pi,\pi]$ which is zero.

EXAMPLE. Let $f(x) = x^3$ on $[-\pi,\pi]$. This is an odd function $(f(-x)+f(x)=0)$ so that it has a sin series: with $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3\sin(nx)\,dx = \frac{-1}{\pi}x^3(-\cos(x)) + 3x^2\sin(x) - 6\sin(x) + 6x\cos(x)|_{-\pi}^{\pi})$ which can be simplified to $b_n = 12(-1)^n/n^3 - 2\pi^2(-1)^n/n$.

EXAMPLE 3. Let $f(x) = 9\cos(6x) + \sin(x)$. This **trigonometric polynomial** is already the Fourier series. The nonzero coefficients are $a_6 = 9, b_1 = 1$.

EXAMPLE 3. Let $f(x) = 1$ on $[-\pi/2,\pi/2]$ and $f(x) = 0$ else. This is an even function $f(-x) - f(x) = 0$ so that it has a cos series: with $a_0 = 1/(\sqrt{2}), a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cos(nx)\,dx = \frac{\sin(nx)}{\pi n}|_{-\pi/2}^{\pi/2} = \frac{2(-1)^m}{\pi(2m+1)}$ if $n = 2m+1$ is odd and 0 else. So, the series is $f(x) = 1/2 + \frac{2}{\pi}(\cos(x)/1 - \cos(3x)/3 + \cos(5x)/5 - ...)$.

Fourier series are useful! For example:

- **Partial differential equations**. PDE's like the wave equation $\ddot{u} = c^2u''$ can be solved by diagonalization (see Friday).
- **Sound** Coefficients a_k form the **frequency spectrum** of a sound f. **Filters** suppress frequencies, **equalizers** transform the Fourier space, **compressors** (i.e.MP3) select frequencies relevant to the ear.
- **Analysis:** $\sum_k a_k\sin(kx) = f(x)$ give explicit expressions for sums which would be hard to evaluate otherwise. The Leibnitz sum $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ...$ is an example.
- **Number theory:** Example: if α is irrational, then the fact that $n\alpha$ (mod1) are uniformly distributed in $[0,1]$ can be understood with Fourier theory.
- **Quantum dynamics:** Transport properties of materials are related to spectral questions for their Hamiltonians. The relation is given by Fourier theory.
- **Crystallography:** X ray Diffraction patterns of a crystal, analyzed using Fourier theory reveal the structure of the crystal.
- **Probability theory:** The Fourier transform $\chi_X = E[e^{iX}]$ of a random variable is called **characteristic function**. Independent case: $\chi_{x+y} = \chi_x\chi_y$.
- **Image formats:** like JPG compress by cutting irrelevant parts in Fourier space.
- **Sound /Movie :** MPG, MP3 files are compressed using ideas from Fourier theory.
- **Filters:** One can filter out frequences using programs like Autotune (see project).

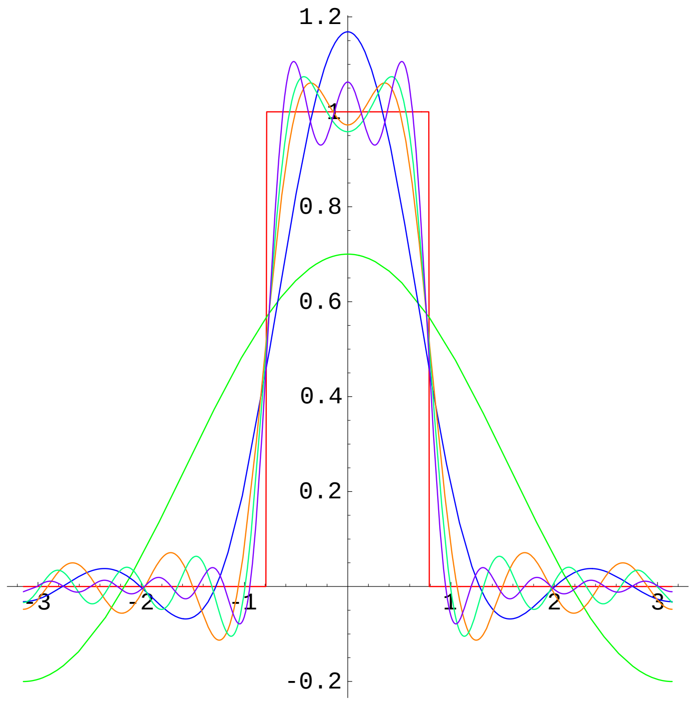
The Parseval identity is $\boxed{||f||^2 = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + b_k^2.}$ Proof. Plug in the series for f. The series converges because any finite sum is bounded above by $||f||^2$ and the sum is increasing.

$f(x) = x = 2(\sin(x) - \sin(2x)/2 + \sin(3x)/3 - \sin(4x)/4 + ...$ has coefficients $f_k = 2(-1)^{k+1}/k$ and so $4(1 + 1/4 + 1/9 + ...) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \, dx = 2\pi^2/3$ or $\boxed{1 + 1/4 + 1/9 + 1/16 + 1/25 + ...) = \pi^2/6}$.

If $f(x) = \sum_k b_k \cos(kx)$, then

$$f_n(x) = \sum_{k=1}^{n} b_k \cos(kx)$$

is an approximation to f. Because $||f - f_k||^2 = \sum_{k=n+1}^{\infty} b_k^2$ goes to zero, the graphs of the functions f_n come for large n close to the graph of the function f. The picture to the left shows an approximation of a piecewise continuous even function in the previous example).



Fourier series are historically interesting because they also tell about the birth of the concept of a function void from analytic expressions: fourier series can approximate functions which are not smooth. The **Greeks** approximation of planetary motion through **epicycles** was an early use of Fourier theory: $z(t) = e^{it}$ is a circle (Aristarchus system), $z(t) = e^{it} + e^{int}$ is an epicycle. **18'th century** Mathematicians like Euler, Lagrange, Bernoulli knew experimentally that Fourier series worked.

Fourier's claim of the convergence of the series was confirmed in the **19'th century** by Cauchy and Dirichlet. For continuous functions, the sum does not need to converge everywhere. However, as the 19 year old **Fejér** demonstrated in his theses in 1900, the coefficients still determine the function if f is continuous and $f(-\pi) = f(\pi)$.

Partial differential equations, to which we come in the last lecture has motivated early research in Fourier theory.

Fourier and Fejer