Homework 3: The number of solutions

This homework is due on Friday, February 2, respectively on Tuesday February 6, 2018.

1 Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$. For each of the vectors $\vec{b}$ given below,

determine whether the system $A\vec{x} = \vec{b}$ has $0, 1$ or ∞ many solutions.

a) $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ b) $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ c) $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ d) $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ e) $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Solution:

We can form the super augmented matrix

$$[A|B] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 4 & 5 & 6 & 2 & 2 & 1 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 & 3 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 7 & 8 & 4 & 2 & 1 & 0 & 0 \\ 5 & 6 & 7 & 8 & 9 & 5 & 1 & 1 & 0 & 1 \end{bmatrix},$$

where B is the matrix whose columns are the proposed vectors $\vec{b}$. If we row reduce the matrix $[A|B]$, we can extract from it the row reduced echelon form $\mathrm{rref}([A|b])$ for each of the possible $\vec{b}$.

$$\mathrm{rref}([A|B]) = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & | & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & | & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Infinitely many solutions.
b) No solution.
c) Infinitely many solutions.
d) Infinitely many solutions.
e) No solutions.

2 Consider the set X of all 2×2 matrices with matrix entries 1 or 2. The probability of a set of matrices Y with some property is the number of matrices in Y divided by the number of matrices in X.
a) What is the probability that the rank of the matrix is 0?
b) What is the probability that the rank of the matrix is 1?
c) What is the probability that the rank of the matrix is 2?

3 As in the previous problem, now also the 2-vector b takes randomly the values 1,2, we can look at all the possible equations $Ax = b$, where A, b are obtained with 1 or 2 entries. The probability space has now 64 elements.
a) What is the probability that the system has a unique solution?
b) What is the probability that the system has no solution?
c) What is the probability that the system has infinitely many solutions?

Solution:
a) The system of equations has a unique solution if the matrix A has full rank, regardless of the value of b. We saw in the previous problem that the probability of this occurring is $\frac{5}{8}$.
b) We have no solution if the rank of $B = [A|b]$ is larger than the rank of A. It happens if $\text{rank}(A) = 1$ and $\text{rank}([A|b]) = 2$. There are 14 cases. The probability is $14/64$. c) There are $64-40-14 = 10$ cases. The probability is $10/64$.

4 Build your own system of equations for three variables or state that there is none. Your system has to have the form $a_{11}x+a_{12}y+a_{13}z = b_1$, $a_{21}x + a_{22}y + a_{23}z = b_2$, $a_{31}x + a_{32}y + a_{33}z = b_3$ with all a_{ij} nonzero.
a) An example with exactly one solution.
b) An example with no solutions.
c) An example where the solution is a plane.

d) An example where the solution is a line.

e) An example where the solution space is three dimensional.

Solution:

a) $x + y + z = 1, x - y + z = 1, x + y - z = 1$

b) $x + y + z = 1, x + y + z = 2, x - y + z = 10$

c) $x + y + z = 1, 2x + 2y + 2z = 2, 3x + 3y + 3z = 3$

d) $x + y + z = 1, 2x + 2y + 2z = 2, x + y - z = 1$

e) there is none. We would need all $a_{ij} = 0$.

5 In a herb garden, the soil has the property that at any given point the humidity is the sum of the neighboring humidities. Samples are taken on a hexagonal grid on 14 spots. The humidity at the four locations x, y, z, w is unknown. Solve the equations $\begin{vmatrix} x = & y+z+w+2 \\ y = & x+w\text{-}3 \\ z = & x+w\text{-}1 \\ w = & x+y+z\text{-}2 \end{vmatrix}$ using row reduction.
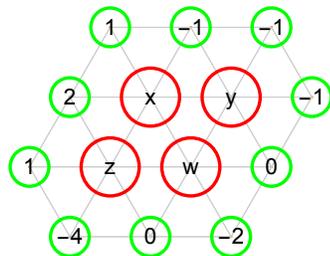
Solution:

We have to find all the solutions to the equations

$$
\begin{vmatrix}
x & - & y & - & z & - & w & = & 2 \\
-x & + & y & & & - & w & = & -3 \\
-x & & & + & z & - & w & = & -1 \\
-x & - & y & - & z & + & w & = & -2
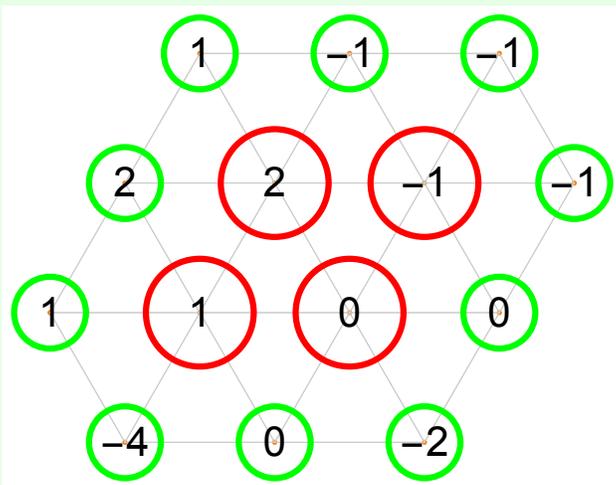\end{vmatrix}
$$

We row reduce the augmented matrix

$$
B = \left[\begin{array}{cccc|c}
1 & -1 & -1 & -1 & 2 \\
-1 & 1 & 0 & -1 & -3 \\
-1 & 0 & 1 & -1 & -1 \\
-1 & -1 & -1 & 1 & -2
\end{array}\right]
$$

and get after a few row reduction steps (left out here)

$$
\mathrm{rref}(B) = \left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{array}\right].
$$

There is exactly one solution. $\boxed{x = 2, y = -1, z = 1, w = 0}$.

Main properties

A system which has a solution is called **consistent**. Otherwise it is called **inconsistent**.

We have a unique solution to $A\vec{x} = \vec{b}$ if and only if rref(A) has a leading 1 in every column and the system is consistent. We have no solution if and only if rref$(A|b)$ has a leading 1 in the last column. In all other cases we have infinitely many solutions.