

## Homework 8: Basis

This homework is due on Wednesday, February 14, respectively on Thursday, February 15, 2018.

- 1 Which of the following sets are linear spaces? Check in each case the three properties characterizing a linear space. Only a brief explanation is needed (can be a picture too):
- a)  $W = \{(x, y, z) \mid x > 0\}$
  - b)  $W = \{(x, y, z) \mid xyz = 0\}$
  - c)  $W = \{(x, y, z) \mid x = 2y = 3z\}$
  - d)  $W = \{(x, y, z) \mid x = y = z + 1\}$
  - e)  $W = \{(x, y, z) \mid x^2 + y^2 - z^2 = 0\}$
  - f)  $W = \{(x, y, z) \mid x, y, z \text{ are rational numbers}\}$
  - g)  $W = \{(x, y, z) \mid x = y = z = 0\}$

### Solution:

- a) No: This space is not a linear space as we can not scale.
- b) No: it is a union of coordinate planes. We can not add
- c) This is a linear space, a line.
- d) This is not a linear space. It does not contain  $(0, 0, 0)$ .
- e) This is not a linear space. We can not scale.
- f) This is not a linear space. We can not scale.
- g) This is a linear space. The trivial space.

- 2
- a) Write the three dimensional space  $x + 2y + 3z + 4t = 0$  as a kernel of a  $1 \times 4$  matrix.
  - b) Write the same plane as the image of a  $4 \times 3$  matrix.
  - c) Find a basis for this space.

**Solution:**

a)  $A = [1, 2, 3, 4]$ .

b)  $A = \begin{bmatrix} -2 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

c)  $\mathcal{B} = \{\underline{v}_\infty, \underline{v}_\epsilon, \underline{v}_\exists\}$ , where  $v_i$  are the columns of the matrix in b).**3** Check whether the given set of vectors is linearly independent

a)  $\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$ . b)  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ . c)  $\left\{ \begin{bmatrix} 3 \\ 16 \end{bmatrix}, \begin{bmatrix} 4 \\ 18 \end{bmatrix}, \begin{bmatrix} 5 \\ 19 \end{bmatrix} \right\}$ .

**Solution:**

a) Write the three vectors as a matrix  $A = \begin{bmatrix} 3 & 3 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$ . Now

row reduce  $A$ . It is  $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ . There are only two leading 1.

The vectors are linearly dependent.

b) Again put the 4 vectors into a matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

Now row reduce to get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are only 3 leading one showing that the four vectors were not linearly independent. There is one free variable and  $[0, 0, 1, -1]^T$  is in the kernel of  $A$ .

c) There are three vectors in  $\mathbb{R}^2$ . They can not be linearly independent as in the row reduction, we can only have 2 leading 1. This implies that there is at least one free variable.

- 4 Find a basis for the image as well as as a basis for the kernel of the following matrices

$$\text{a) } \begin{bmatrix} 7 & 0 & 7 \\ 2 & 3 & 8 \\ 9 & 0 & 9 \\ 5 & 6 & 17 \end{bmatrix}, \text{ b) } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}. \text{ c) } \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

**Solution:**

a) Row reduction gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see two Pivot columns with leading 1 and one redundant column with a free variable  $s$ . Write  $x = -s, y = -2s, z = s$

to get the kernel spanned by  $[-1, -2, 1]^T$ . Now  $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$

is a basis for the kernel and the two original Pivot columns are

a basis  $\mathcal{B} = \left\{ \begin{bmatrix} 17 \\ 2 \\ 9 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 6 \\ 2 \end{bmatrix} \right\}$  for the image.

b) This problem is similar to part (a). Row reduction gives

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ , so we have the same kernel as in (a). The image

has the basis  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ .

c) Row reduction gives matrix  $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . This means the

kernel is spanned by  $[1, 0, -1, 0]^T, [0, 1, 0, -1]^T$ . The image is spanned by the columns, which are  $[1, 0, 1]^T, [0, 1, 1]^T$ .

- 5 The orthogonal complement of a subspace  $V$  of  $R^n$  is the set  $V^\perp$  of all vectors in  $R^n$  that are perpendicular to every single vector

in  $V$ . Find a basis for the orthogonal complement in each case:

- a) The line  $L$  in  $R^5$  spanned by  $\begin{bmatrix} 1 & 2 & 2 & 1 & 1 \end{bmatrix}^T$ , (If  $v$  is a row vector  $v^T$  denotes the corresponding column vector).
- b) The plane  $\Sigma$  in  $R^4$  spanned by  $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$  and  $\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^T$ .
- c) The space  $V = \{(0, 0)\}$  in the two-dimensional plane  $R^2$ .

**Solution:**

a) Let  $A = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 \end{bmatrix}$ . We seek to solve  $Ax = 0$ , as  $V^\perp$  is the kernel of  $A$ . We have four free parameters- which we can take to be the second through fifth entries in the vector. This gives us the basis  $\mathcal{B} = \left\{ \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} -2 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \right\}$ .

b) Similar to before, we want to solve  $Ax = 0$  for  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ . Row reduction gives  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ .

There are clearly two free parameters, which we denote  $s, t$ . Setting  $z = s, w = t$ , we have  $x = -t, y = -s$ . A basis for the kernel is  $\mathcal{B} = \left\{ \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}^T \right\}$ .

c) Every vector is orthogonal to the zero vector, so  $V^\perp$  is the entire plane,  $\{e_1, e_2\}$ .

**Basis**

$V$  is a **linear space** if  $0$  is in  $V$ , if  $v + w$  is in  $V$  for all  $v, w$  in  $V$  and if  $\lambda v$  is in  $V$  for every  $v$  in  $V$  and every  $\lambda$  in  $\mathbb{R}$ . Examples: kernels  $V = \ker(A)$  or images  $V = \text{im}(A)$  are linear spaces. If  $V, W$  are linear spaces and  $V$  is a subset of  $W$ , then  $V$  is called a **linear subspace** of  $W$ . A line through the origin for example is a linear subspace of  $\mathbb{R}^3$ . A set  $\mathcal{B}$  of vectors  $\{v_1, \dots, v_n\}$  **spans**  $V$  if every  $v \in V$  is a sum of vectors in  $\mathcal{B}$ . A set  $\mathcal{B}$  is linear independent if  $a_1 v_1 + \dots + a_n v_n = 0$  implies  $a_1 = \dots = a_n = 0$ . It is a **basis** of  $V$  if it both **spans**  $V$  and is linearly independent. Example: the standard basis vectors  $\{e_1, \dots, e_n\}$  form a basis of  $\mathbb{R}^n$ . How do we determine whether a set of vectors is a basis of  $\vec{R}^n$ ? Place the vectors of  $\mathcal{B}$  as columns in a matrix  $A$ , then row reduce  $A$ . If every column of a matrix has a leading 1, then the set of column vectors  $\mathcal{B}$  are linearly independent and the kernel of  $A$  is  $\{0\}$ . How do we determine whether a set of vectors is linearly independent? Place the vectors as columns of a matrix and row reduce. If there is no free variable, then we have linear independence. Example: three vectors in  $\mathbb{R}^3$  are linearly independent if they are not in a common plane.