

Homework 11: Linear spaces

This homework ¹ is due on Friday, February 23, respectively on Thursday, February 22, 2018.

- 1 Which of the following spaces are linear spaces?
- a) All 4×4 matrices for which the diagonal entries are all zero.
 - b) All 4×4 matrices for which the product of all matrix entries is zero.
 - c) All polynomials of degree exactly 4.
 - d) All 2×2 projection dilation matrices $A = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$.
 - e) All 2×2 matrices A for which $A^2 = 0$.
 - f) All functions f in $C(\mathbb{R})$ for which $f(5) = 0$.
 - g) All functions f in $C^\infty(\mathbb{R})$ for which $f''(0) = f'(1)$.
 - h) All functions f in $C^\infty(\mathbb{R})$ for which $f(0)^2 + f''(2) = 0$.

¹While not much seems to happen in this lecture we actually make a giant step. The language of linear algebra is pushed into realms where it originally was not used like to spaces of functions used in calculus. You can now look at a spread sheet or picture or movie or a table in a relational database and see it as a vector in a linear space.

Solution:

a) is a linear space. Check the three conditions: the zero matrix is there, the matrix elements add up.

b) is not a linear space. The sum of such matrices like $A = \text{Diag}(1, 2, 3, 0)$ and $B = \text{Diag}(0, 1, 2, 3)$ is no more of this type.

c) is not a linear space, because it does not contain the zero function.

d) is not a linear space. If you multiply a by a factor 2, the resulting matrix is not scaled by 2.

e) is not a linear space. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

They are both in the linear space but the sum is not there.

f) is a linear space.

g) is a linear space. Check all properties.

h) is not a linear space.

2 Find a basis for the space of

a) all 2×2 rotation dilation matrices $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

b) all 2×2 reflection dilation matrices $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$.

c) all 2×2 horizontal shears. (Trick question! Why?)

d) all the 2×2 matrices for which Ae_1 is parallel to e_1 .

e) all the diagonal 3×3 matrices.

f) all the 2×2 dilation matrices.

Solution:

$$\text{a) } \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}.$$

$$\text{b) } \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

c) This is not a linear space!

$$\text{d) } \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

$$\text{e) } \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

$$\text{f) } \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

3 Find a basis for all the 2×2 matrices A for which

$$A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution:

The condition on the matrix A is equivalent to $A(e_1 + e_2) = 0$, which means that both rows must be orthogonal to $e_1 + e_2$ (and are therefore parallel to each other). This gives us a basis of

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}.$$

4 A function is called **even** if $f(-x) = f(x)$ for all x . A function is called **odd** if $f(-x) = -f(x)$ for all x . Find a basis:

a) for all the even polynomials in $P_4(\mathbb{R})$

b) for all the odd polynomials in $P_4(\mathbb{R})$.

c) for all polynomials f in $P_4(\mathbb{R})$ for which $f(0) = 0$.

Solution:

a) $\mathcal{B} = \{1, x^2, x^4\}$.

b) $\mathcal{B} = \{x, x^3, \}$.

c) $\mathcal{B} = \{x, x^2, x^3, x^4\}$.

- 5 a) Find a concrete 3×3 matrix A with no zeros for which $A\vec{v} = \vec{0}$, where $\vec{v} = [1 \ 2 \ 3]^T$.
- b) Let V be the set of 3×3 matrices for which \vec{v} is in the kernel. Is this a linear space?

Solution:

a) This means that all of the rows are orthogonal to \vec{v} . One

example is $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$.

b) Yes, think about each condition.

Linear space

A set V in which we can add, scale and which contains a 0 is a **linear space**. To check that V is a linear space, verify (i) 0 is in X , (ii) if x and y are both in X , then $x + y$ is in X , (iii) if x is in X , then λx is in X for all constants c . There are three important classes of linear spaces: first the subspaces of R^n as treated before, the second class is the set $M(n, m)$ of all $n \times m$ matrices. Finally, there is the class $C(R)$ for all continuous functions on the real line. It contains the linear space $C^\infty(R)$ for all smooth functions, functions which can be differentiated arbitrary often. We also write $P_n(R)$ for the set of polynomials of degree less or equal to n . It has dimension $n + 1$. In order to check whether a subset of functions or matrices or vectors in R^n are a linear space, check the three things (i),(ii),(iii). In the case of functions, 0 is the function which is 0 for all x .