

Homework 15: Data fitting

This homework is due on Wednesday, March 7, respectively on Thursday, March 8, 2018.

- 1 a) Find the least square solution x^* of the system $Ax = b$ with

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 2 & -4 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 125 \\ 125 \\ 125 \end{bmatrix}.$$

- b) What is the matrix P which projects on the image of A ?

Solution:

a) To get the least square solution to $Ax = b$ first get $A^T A = \begin{bmatrix} 9 & -8 \\ -8 & 21 \end{bmatrix}$. We obtain $x^* = (A^T A)^{-1} A^T b = \begin{bmatrix} 81 \\ 13 \end{bmatrix}$.

b)

$$P = \begin{bmatrix} 61 & 48 & 40 \\ 48 & 89 & -30 \\ 40 & -30 & 100 \end{bmatrix} / 125 .$$

- 2 Find the function $y = f(x) = 1 + ax + bx^3$, which best fits the data

x	y
-1	5
2	4
-2	8

Solution:

We can think of x and x^3 as separate variables, giving us triplets of variables from the data: $(-1, 4; 4)$, $(2, 8; 3)$, $(-2, -8; 7)$. We wish to find the least square solution to the system of equations

$$\begin{aligned} -a - b &= 4 \\ 2a + 8b &= 3 \\ -2a - 8b &= 7 \end{aligned}$$

which can be rewritten in matrix form $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} -1 & -1 \\ 2 & 8 \\ -2 & -8 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}.$$

We have $A^T A = \begin{bmatrix} 9 & 33 \\ 33 & 129 \end{bmatrix}$ and $A^T b = \begin{bmatrix} -12 \\ -36 \end{bmatrix}$. We get the least square solution with the formula

$$x = (A^T A)^{-1} A^T b = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

The best fit is the function $f(x) = 1 - 5x + x^3$.

3 A curve of the form

$$y^2 = x^3 + ax + b$$

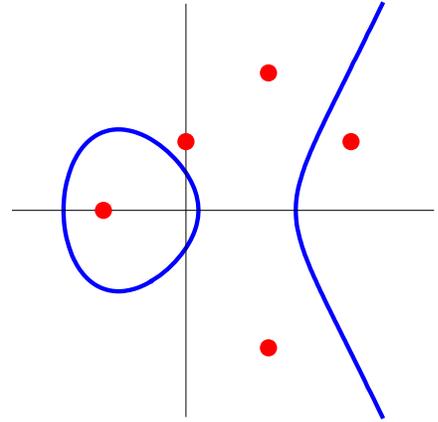
is called an **elliptic curve** in Weierstrass form. Elliptic curves are important in cryptography. Use data fitting to find the best parameters (a, b) for an elliptic curve given the following points:

$$(x_1, y_1) = (1, 2)$$

$$(x_2, y_2) = (-1, 0)$$

$$(x_3, y_3) = (2, 1)$$

$$(x_4, y_4) = (0, 1)$$



Solution:

Setting up the equations $b + ax_i + x_i^3 = y_i^2$ gives the system

$$\begin{aligned}a + b &= 3 \\-a + b &= 1 \\2a + b &= -7 \\0a + b &= 1\end{aligned}$$

which can be written as $A\vec{x} = \vec{b}$, where $\vec{x} = [a, b]^T$ and $\vec{b} = [3, 1, -7, 1]^T$. The least square solution is given by $(A^T A)^{-1} A^T \vec{b}$. One has $A^T A = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$ and therefore

$(A^T A)^{-1} = \frac{1}{20} \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}$. We also have $A^T \vec{b} = \begin{bmatrix} -12 \\ -2 \end{bmatrix}$, so

$(A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} -\frac{11}{5} \\ \frac{3}{5} \end{bmatrix}$. The best fit is given by the Elliptic curve $y^2 = x^3 - \frac{11}{5}x + \frac{3}{5}$.

We leave the comfort zone and look at extreme cases.

a) Analyze the best linear fit $f(x) = ax + b$ for the three data points $(1, 1), (1, 2), (1, 3)$.

4 b) To find the best linear $f(x) = a + bx$ for the four data points $(1, 1), (1, 1), (2, 2), (4, 7)$ with $f(x) = a + bx$, we shorten the climb by taking the points $(1, 1), (2, 2), (4, 7)$ as one point was redundant. Do we get the same?



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Solution:

a) Write down the system $a + b = 1, a + b = 2, a + b = 3$. The matrix A has a kernel and $A^T A$ is not invertible. What happens is that there are many solutions to this problem.

b) The two solutions are different. In the first case, the best solution is $y = 2x - 5/4$. In the second case case, $y = 29/14x - 1.5$.

5 Find the circle $a(x^2 + y^2) + b(x + y) = 1$ which best fits the data

x	y
0	1
-1	0
1	-1
1	1

In other words, find the least square solution for the system of equations for the unknowns a, b which aims to have all 4 data points (x_i, y_i) on the circle.

Solution:

To get system of linear equations $Ax = b$, plug in the data

$$a + b = 1$$

$$a - b = 1$$

$$2a = 1$$

$$2a + 2b = 1.$$

This can be written as $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:

This gives us $A^T A = \begin{bmatrix} 10 & 4 \\ 4 & 6 \end{bmatrix}$. Now, we compute

$$(A^T A)^{-1} = \frac{1}{22} \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$$

and

$$A^T b = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

We obtain $x^* = (A^T A)^{-1} A^T b = \frac{1}{11} \begin{bmatrix} 7 \\ -1 \end{bmatrix}$, so the best fit is $\frac{7}{11}(x^2 + y^2) - \frac{1}{11}(x + y) = 1$.

Given a system $Ax = b$. Any solution of $(A^T A)x = A^T b$ is called a **least square solution** (these always exist). (Reason: solve $A^T(Ax - b) = 0$ for x , assuring that $Ax - b$ is perpendicular to $\text{im}(A)$.) The least square solution is unique if A has a trivial kernel. In that case $x = (A^T A)^{-1}A^T b$. The matrix $A(A^T A)^{-1}A^T$ is now the projection matrix onto $\text{im}(A)$. If the columns of A are orthonormal, this simplifies to $P = AA^T$.