Homework 25: Differential equations II

This homework is due on Monday, April 9, respectively on Tuesday, April 10, 2018.

1 a) Find the solution to the differential equation $\frac{d^2x}{dt^2} = -x$ with initial conditions $x(0) = 5, \frac{dx}{dt}(0) = 0$ by writing the initial condition $\vec{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ as a linear combination of eigenvectors $[i, 1]^T$ and $[i, -1]^T$ of A, where

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

is the system written in vector form using 2×2 matrix A.

Solution:

We can write $\vec{v}'(t) = Av(t)$ with $v(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues of A are $i, -i$ with eigenvectors $v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} i \\ -1 \end{bmatrix}$. The initial condition is a sum of the eigenvectors $5/(2i)v_1 + (5/2i)v_2$ so that

$$v(t) = (5/(2i))e^{it}v_1 + (5/(2i))e^{-it}v_2 \ .$$

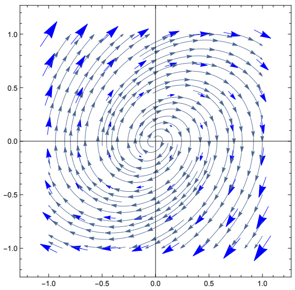
We can rewrite this as

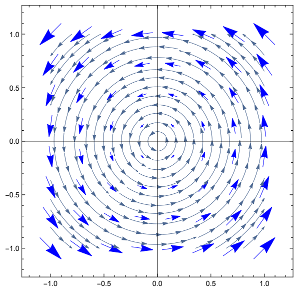
$$v(t) = \begin{bmatrix} 5\cos(t) \\ 5\sin(t) \end{bmatrix} .$$

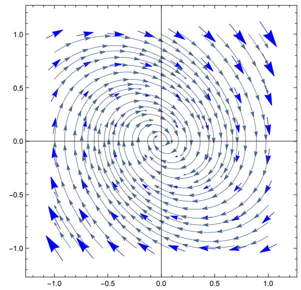
2 Match the differential equations $\frac{dx}{dt} = Ax$ with the phase portraits.

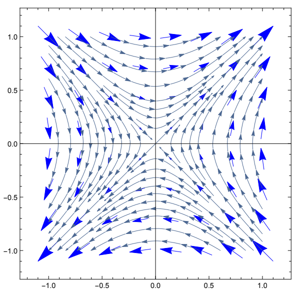
i) $A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ ii) $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ iii) $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, iv) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

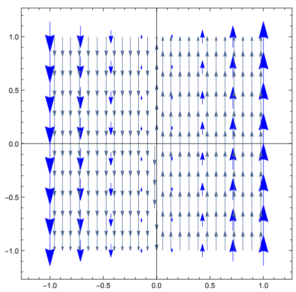
v) $A = \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}$ vi) $A = \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}$

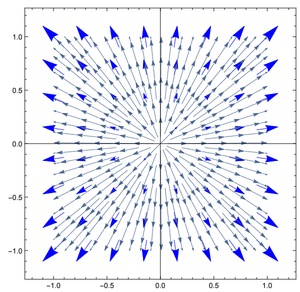
a) 

b) 

c) 

d) 

e) 

f) 

Solution:

The eigenvalues determine the match in each case a) Here we have an outward spiral, so we must have positive real part and nonzero imaginary part, giving us v).

b) In this case, the distance from the origin is constant, so the eigenvalues are purely imaginary, giving us ii).

c) This is an inward spiral, so we must have negative real part and nonzero imaginary part, giving us vi).

d) Here we have a semistable system, so we will have two real eigenvalues, with one positive and one negative. We deduce that this is iii).

Solution:
e) This portrait arises from a matrix with a zero eigenvalue. This is i).
f) This diagram points away from the origin everywhere, telling us that the matrix has two real eigenvalues, both of which are positive, giving us iv).

3 Determine the stability of the systems

$$a)v'(t) = \begin{bmatrix} 7 & 11 & 23 & 8 \\ 9 & -7 & 5 & -1 \\ -4 & -2 & 3 & 4 \\ 5 & -4 & 1 & -3 \end{bmatrix} v(t), \quad b)v'(t) = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 9 & -7 & 0 & 0 \\ -1 & -1 & -2 & -4 \\ 1 & -1 & 4 & -2 \end{bmatrix} v(t)$$

Solution:
a) This system is unstable; the matrix has trace zero, so at least one eigenvalue has to have a nonnegative real part.
b) This system is stable; all eigenvalues $(-3, -1, -2 \pm 4i)$ have negative real part.

4 To solve the fourth order equation $w''''(t) = w(t)$, we write it as a system of first order differential equations of the form $\vec{v}'(t) = A\vec{v}(t)$ using $\vec{v}(t) = (x(t), y(t), z(t), w(t))$ where $w'(t) = z(t), z'(t) = y(t), y'(t) = x(t), x'(t) = w(t)$ and A is a 4×4 matrix. Write down a general closed form solution formula $\vec{v}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_4 e^{\lambda_4 t}\vec{v}_4$, where c_1, c_2, c_3, c_4 are parameters. Is this system stable or unstable?

Solution:
This is the cyclic matrix A we have seen in a previous homework. The eigenvalues are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i, \lambda_4 = -i$. The system is unstable, as the eigenvalue 1 is present.

5 True or false? Give short explanations
a) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = A^2x$ is stable.
b) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = (A - 2I_n)x$ is stable.
c) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = A^Tx$ is stable.
d) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = -Ax$ is stable.
e) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = 3Ax$ is stable.

Solution:
a) False: The eigenvalues of A^3 are λ^2 if λ is an eigenvalue of A.
b) True; if λ has negative real part, then so does $\lambda - 2$. c) True; the matrices A and A^T have the same eigenvalues.
d) False; the eigenvalues of $-A$ are the negatives of the eigenvalues of A. In particular, the real parts will have the opposite sign.
e) True; if λ is an eigenvalue, then 3λ is an eigenvalue of $3A$ and if the real part of $\lambda = a + ib$ is positive, then the real part $a/(a^2 + b^2)$ of $1/\lambda$ is positive too.

Differential Equations II

$\frac{d^2x}{dt^2} = -k^2x(t)$ is called **harmonic oscillator**. It has solutions $x(t) = a\cos(kt) + b\sin(kt)$, where a, b depend on initial conditions. It becomes a system $\frac{dx}{dt} = y(t), \frac{dy}{dt} = -k^2x(t)$. In general, for a $n \times n$ matrix A, the system $v' = Av$ is **stable** if all eigenvalues satisfy $\mathrm{Re}(\lambda_j) < 0$.