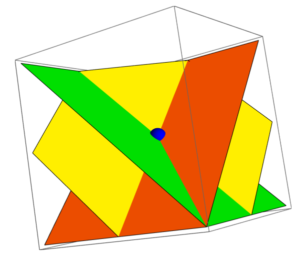
LINEAR ALGEBRA

MATH 21B



Linear Equations

1.1. A collection of linear equations is called a **system of linear equations**. Example

$$\left| \begin{array}{ll} x+y+z & =1 \\ x+y & =2 \\ x+z & =3 \end{array} \right| .$$

These are $n = 3$ equations for $m = 3$ **unknowns** x, y, z. **Linear** means that no nonlinear terms like $x^2, x^3, xy, yz^3, \sin(x)$ appear. A system is called **consistent** if there exists a solution. It is called **inconsistent** if there is no solution. We write this also in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

or shortly as $A\vec{x} = \vec{b}$. Vectors are usually **column vectors**. Think of the matrix A as built from its columns. The matrix is is a $n \times m = 3 \times 3$ matrix. There are m columns and n **rows**.

1.2. While a computer solves this with a built in algorithm

$$\text{Solve}[\{x+y+z==1, x+y==2, x+z==3\}, \{x,y,z\}] .$$

we should be able to do that by manipulating an **augmented** 3×4 **matrix**. In this case all variables are **leading variables**. There is no **free variable**.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 \end{bmatrix} .$$

1.3. There is a geometric aspect. Each equation is a **plane** in space. The solution is the intersection of these planes. If three planes do not intersect in a common point, there is no solution.

A tomography example

1.4. Computer aided tomography CAT uses magnetic resonance imaging MRI to see inside a body. Assume a molecule with 4 atoms is excited with energy intensity a, b, c, d. We measure the spin echo in 4 different directions. $3 = a + b$,$7 = c + d$,$5 = a + c$ and $5 = b + d$. What is a, b, c, d? Solution: $a = 2, b = 1, c = 3, d = 4$. However, also $a = 0, b = 3, c = 5, d = 2$ solves the problem. This system is **consistent** but it has not a **unique solution**, even so there are 4 equations and 4 unknowns. In this case there are 3 leading variable and one free variable.

History

1.5. The history of linear algebra is more than 4000 years old. Around 2000 BC, the **Babylonians** solved single equations. From 250BC is the Archimedes cattle problem, a system of equations for 8 unknowns and 7 equations. This has infinitely many solutions but the problem asks for integer solutions, a Diophantine problem. In China, the **Nine chapters of mathematical art** appeared around 200BC-200CE and contains linear algebra problems. Also the in 300 AD authored **Bakshali manuscript** from India, famous also for the first appearance of 0 contains some linear algebra. Around 800, Al Khawarizmi systematically produced geometric solutions to linear and quadratic equations. In 1750, Cramers found solution formulas using determinants. Around 1800 Gauss formalized elimination and least square solutions.

1.6. Here is a riddle found near Bagdad on a tablet found in the **Tell Harmal excavations**. It is a single equation for a single unknown. [1]

If somebody asks you thus: if I add to the two thirds of my two thirds a hundred aq, of barley, the original quantity is , summed up. How much is the original quantity?

This translates to the problem

$$\frac{4}{9}x + 100 = x$$

1.7. Here is a riddle from the Bakshali manuscript:

One person possesses seven Asava horses, another has nine Haya horses, and another ten camels. Each gives two animals, one to each of the others. They are then equally well off. Find the price of each animal and the total value of the animals possesses by each person.

To rewrite this mathematically, let x be the prize of a Asava horse, y the prize of a Haya horse and z the prize of a camel. The first person has 5 Asava horse, the second 7 Haya horses and the third has 8 camels. Let t be the total. We actually have 3 equations and 4 unknowns here:

$$\begin{aligned} 5x + y + z &= t \\ 7y + x + z &= t \\ 8z + y + x &= t \end{aligned}$$

1.8. An other example from China: Nine Chapters of Mathematical Art, from the Han Dynasty, 200 BC:

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned}$$

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[1] Source: George Gheverghese Joseph: The crest of the peacock: Non-European roots of Mathematics