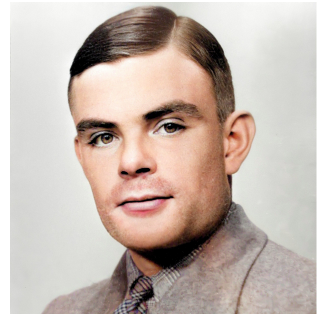
LINEAR ALGEBRA

MATH 21B



Row Reduction

2.1. We have seen that a system of linear equations $A\vec{x} = \vec{b}$ with n equations of m unknowns can be solved by manipulating the augmented matrix $B = [Ab]$ in which the vector b has been appended as a new column. To solve the system, we **row reduce** the matrix B until we can read off the solution. Let us formalize this.

2.2. Gauss-Jordan elimination is an algorithm which produces from a matrix B a **row reduced matrix** $\text{rref}(B)$. Here are the three moves you are allowed:

Subtract a row from another row	**Scale a row**	**Swap two rows**

If B came from a system $A\vec{x} = \vec{b}$, all these operations preserve the solution space.

2.3. A **leading ones** ① is a matrix entry 1 which is the first non-zero entry in a row. The corresponding variable is a **leading variable**. The corresponding column is a **leading column**. A row with a leading 1 is a **leading row**. A matrix is in **row reduced echelon form** if

Every non-zero row has a ① and so is leading.
① is the only non-zero entry in a leading column.
Every row above a leading row has ① to the left.

"Leaders want to be first, do not like other leaders in their column and want every row above equipped with a leader left to them."

2.4. Here is an example on how to do row reduction

$\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 8 & 10 & 12 \\ 1 & 2 & 3 & 4 & 4 \end{bmatrix}$	$\begin{array}{l} \to \text{swap } R_3 \\ \\ \to \text{swap } R_1 \end{array}$	$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 4 & 6 & 8 & 10 & 12 \\ 3 & 4 & 5 & 6 & 7 \end{bmatrix}$	$*1/2$
$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{bmatrix}$	$\begin{array}{l} \\ -R_1 \\ -R_2 \end{array}$	$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{array}{l} \\ -R_1 \\ -R_2 \end{array}$
$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{array}{l} +2R_2 \\ *(-1) \\ \end{array}$	$\begin{bmatrix} \textcircled{1} & 0 & -1 & -2 & -3 \\ 0 & \textcircled{1} & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	

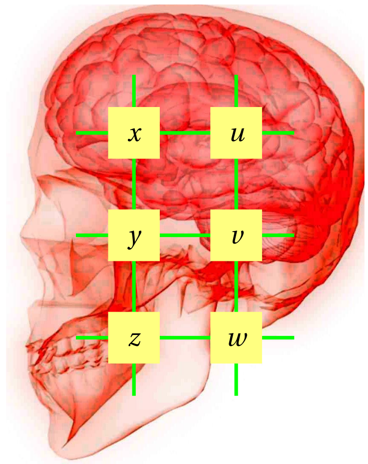


FIGURE 1. Magnetic Resonance Imaging is a radiology imaging technique that avoids radiation exposure to the patient. Solving a system of equations allows to compute the actual densities and so to do the magic of "seeing inside the body".

2.5. The system of equations

$$\left| \begin{array}{ccccccccccccc} x & & & & & + & u & & & & & = & 3 \\ & & y & & & & & + & v & & & = & 5 \\ & & & & z & & & & & + & w & = & 9 \\ x & + & y & + & z & & & & & & & = & 8 \\ & & & & & & u & + & v & + & w & = & 9 \end{array} \right|$$

is an other **tomography** problem. Lets solve it.

$$\begin{bmatrix} \textcircled{1} & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & \textcircled{1} & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 9 \\ 1 & 1 & 1 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & \textcircled{1} & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 1 & 9 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & -1 & -1 & -6 \\ 0 & \textcircled{1} & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that v and w are free variables. We write $v = t$ and $w = s$. Then just solve for the variables:

$$\begin{array}{rcl} x & = & -6 + t + s \\ y & = & 5 - t \\ z & = & 9 - s \\ u & = & 9 - t - s \\ v & = & t \\ w & = & s \end{array}$$

2.6. Which matrices are in row reduced echelon form? What is the **rank**, the number of $\textcircled{1}$ in each case?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$