

# LINEAR ALGEBRA

MATH 21B



## LINEAR TRANSFORMATIONS

**3.1.** A transformation  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is called **linear** if there is a  $n \times m$  matrix  $A$  such that  $T(\vec{x}) = A\vec{x}$ . The vector  $\vec{x}$  is in the **domain**  $\mathbb{R}^m$ .  $A\vec{x}$  is in the **codomain**  $\mathbb{R}^n$ .

**3.2.** Linear transformations are characterized by three properties:

**Theorem:**  $T$  is linear if and only for all  $\vec{x}, \vec{y}$  and  $k$  we have:

$$\boxed{\text{i) } T(\vec{0}) = \vec{0}}$$

$$\boxed{\text{ii) } T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})}$$

$$\boxed{\text{iii) } T(k\vec{x}) = kT(\vec{x})}$$

In other words,  $T$  is compatible with 0, addition and multiplication. Geometrically,  $T$  preserves the origin, parallelograms and length ratios of parallel vectors.

**3.3.** Some texts omit the first condition  $T(0) = 0$  because it follows from *iii*) and setting  $k = 0$ , but it is easier to start checking this condition first. If a transformation does not satisfy *i*) you do not have to check further. To see for example that  $T(\vec{x}) = 2\vec{x} + 3$  is not linear, just plug in  $\vec{x} = \vec{0}$  and see  $T(\vec{0}) = 3$ . You are done and do not have to investigate further. It looks like omitting *i*) is more effective but the contrary is the case. By the way all three conditions could be placed into one  $T$  is linear if and only if  $T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$  for all  $a$  and  $b$ . But that is even less effective to check.

**3.4.** To verify the statement, first check that a transformation  $T(\vec{x}) = A\vec{x}$  satisfies all three condition. In order to see the converse, we need to be able to **construct the matrix**  $A$ . We construct the matrix  $A$  column by column.

**3.5.** Consider the basic vectors (they are so basic that one officially calls them basic!)

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}, \dots, \vec{e}_m = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \end{bmatrix}.$$

**Theorem:** If  $T$  is linear, the  $k$ 'th column of the matrix  $A$  is  $T(\vec{e}_k)$ .

**3.6.** Say, you want to find the matrix which belongs to the linear transformation rotating in space around the axes  $x = y = z$  counter clockwise when looking from from  $(1, 1, 1)$  to  $(0, 0, 0)$ . Since  $\vec{e}_1$  maps to  $T(\vec{e}_1) = \vec{e}_2$  and  $\vec{e}_2$  maps to  $T(\vec{e}_2) = \vec{e}_3$  and  $\vec{e}_3$

lands in  $T(\vec{e}_3) = \vec{e}_1$ , we can fill in the columns of  $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

**3.7.** To see for example what the effect of  $T(\vec{x}) = A\vec{x}$  with  $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is, we look at the column vectors. They have the same length and are perpendicular to each other. The transformation is a rotation dilation about the  $z$ -axes as rotation.

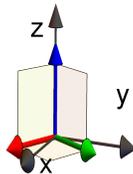


FIGURE 1. A rotation dilation in space. We link the transformation with the matrix by looking at the image of the basis vectors.

**3.8. Example:**  $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y + x \\ 0 \\ x^2 - y \end{bmatrix}$  is not linear because the zero vector is not mapped into the zero vector.

**3.9.**  $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + 2y \\ y + z \\ 0 \end{bmatrix}$ . Indeed, we can construct a matrix  $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  and see  $T(\vec{x}) = A\vec{x}$ .

**3.10.**  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} |x| - 3y \\ y \end{bmatrix}$  is not linear. While  $T(\vec{0}) = \vec{0}$ , the second and third conditions both fail. Take for example  $k = -3$  and  $\vec{v} = \vec{e}_1$ .

**3.11.**  $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  is linear with  $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

**3.12.** Find the matrix of the linear transformation  $T(x, y, z) = x + y + z$ ! Answer: This maps  $\mathbb{R}^3$  to  $\mathbb{R}^1$ . To get  $A$  find the columns. Because  $T(\vec{e}_1) = 1$  and  $T(\vec{e}_2) = 1$  and  $T(\vec{e}_3) = 1$ , we have a  $1 \times 3$  matrix  $A = [1, 1, 1]$ .

**3.13.** The find matrix of the transformation  $T(x, y) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ , first realize  $T$  is a map from  $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ .  $T(1, 0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $T(0, 1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . The matrix is  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ .