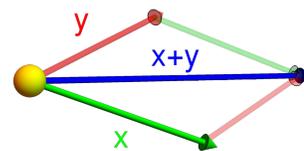


# LINEAR ALGEBRA

MATH 21B



## LINEAR COMBINATIONS

**4.1.** Recall that a **transformation**  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  is a rule, which assigns to every  $\vec{x}$  in the **domain**  $\mathbb{R}^m$  a vector  $\vec{y} = T(\vec{x})$  in the **co-domain**  $\mathbb{R}^n$ . It is **linear** if there is a  $n \times m$  matrix  $A$  such that  $T(\vec{x}) = A\vec{x}$ . Linear transformations are compatible with addition, scalar multiplication and the zero element. One of the most important facts in linear algebra is the **column picture**:

**Theorem:** In the matrix  $A$  belonging to  $T$

$$A = \left[ \begin{array}{c|c|c|c} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & \cdots & | \end{array} \right],$$

the column vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  are the images  $\vec{v}_k = T(\vec{e}_k)$  of the **stan-**

**dard basis vectors**  $\vec{e}_1 = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \dots, \vec{e}_k = \begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{bmatrix}, \dots, \vec{e}_m = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$ .

We can express this as

$$A \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_m \end{bmatrix} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m = \vec{b}.$$

Every vector the range of  $T$  is expressed as a scaled sum of column vectors of  $A$ .

**4.2.** A sum  $\vec{b} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$  is called a **linear combination** of  $\vec{v}_1, \dots, \vec{v}_m$ . For example,

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a linear combination of the vectors  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

**4.3.** We say the vectors  $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$  **span**  $\mathbb{R}^m$  if every  $\vec{b} \in \mathbb{R}^m$  can be expressed as a linear combination of vectors in  $\mathcal{B}$ . This means that the system of linear equations

$$A\vec{x} = \vec{b}$$

has a solution for all  $\vec{b}$ . We say the vectors  $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$  are **linearly independent** if there are no free variables in the row reduced echelon form of  $A$ . If  $\mathcal{B}$  are both linearly independent and span, they form a **basis**.

**4.4.** The basis vectors  $\vec{e}_1, \dots, \vec{e}_m$  span  $\mathbb{R}^m$  because written as a combination of these vectors. For example,

$$\vec{v} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

**4.5.** Here is a useful fact:

**Theorem:** If  $m$  vectors  $\mathcal{B}$  span  $\mathbb{R}^m$ , they are also linearly independent.

The justification of this statement is that spanning means that there are  $m$  leading 1 after row reduction of the matrix  $A$  with columns  $\vec{v}_k$ . But as there are  $m$  columns, this means that there are no free variables and the vectors are also linearly independent.

**4.6. Example:** The vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

form a basis of  $\mathbb{R}^5$ . To see this, row reduce its matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

**4.7.** A basis is useful because if  $T$  is a linear transformation and  $T(\vec{v}_1), \dots, T(\vec{v}_m)$  are all known, we also know  $T(\vec{v})$  for a linear combination  $\vec{v} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$  of  $\vec{v}_1, \dots, \vec{v}_m$ . We have then

$$T(c_1\vec{v}_1 + \dots + c_m\vec{v}_m) = c_1T(\vec{v}_1) + \dots + c_mT(\vec{v}_m).$$

Linear transformations preserve linear combinations.

**4.8.** Summary of the most important things seen here:

To find the matrix  $A$  of a linear transformation  $T$ , look at the image  $\vec{v}_k = A\vec{e}_k$  of the standard basis vectors  $\vec{e}_k$  and use them to build up the columns  $\vec{v}_k$  of  $A$ .

A collection  $\mathcal{B}$  of  $m$  vectors  $\vec{v}_k$  in  $\mathbb{R}^m$  is a **basis** of  $\mathbb{R}^m$  if it spans  $\mathbb{R}^m$  and is linearly independent. Spanning means that  $A$  has  $m$  leading  $\textcircled{1}$  and linear independence means that there are no free variables in  $\text{rref}(A)$ . A collection of  $m$  vectors in  $\mathbb{R}^m$  which span are automatically linearly independent.