LINEAR ALGEBRA

MATH 21B

Matrix product

6.1. If $T(\vec{x}) = A \cdot \vec{x}$ is a linear transformation from $\mathbb{R}^m \to \mathbb{R}^n$ if $S(\vec{x}) = B \cdot \vec{x}$ is a linear transformation from $\mathbb{R}^p \to \mathbb{R}^m$, then $T(S(\vec{x})) = A \cdot (B\vec{x})$ is a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^n$. Its matrix is $A \cdot B$ and called the **matrix product**.

6.2. If A is a $n \times m$ matrix and A is a $m \times p$ matrix, then AB is defined as the $n \times p$ matrix with entries $(BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}$. It represents a linear transformation from $R^p \to R^n$, where first B is applied as a map from $R^p \to R^m$ and then the transformation A from $R^m \to R^n$. Here we see an example of a multiplication of a 3×5 matrix with 5×2 matrix resulting in a 3×2 matrix. When computing the entries of the result, every row in the first matrix gets dotted with a column of the second matrix. If A, B are $n \times n$ matrices, then $A \cdot B$ is a $n \times n$ matrix. We can therefore define the n'th power $A^n = A \cdot A \cdot \cdots \cdot A$ like in the case of numbers. We can also look at terms like $A^2 + A - 1$.

$$\begin{array}{|ccccc|} \hline 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \begin{array}{|cc|} \hline 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ \hline \end{array} = \begin{bmatrix} 5 & 10 \\ 2 & 4 \\ 5 & 10 \end{bmatrix}$$

6.3. Example 1 If B is a 3×4 matrix, and A is a 4×1 matrix,then BA is a 3×1 matrix. You see that the multiplication of a matrix with a vector is a special case as we can identify $m \times 1$ matrices as vectors in $\mathbb{R}^m$. If A was a $n \times m$ matrix, then the result $A\vec{v}$ is a $n \times 1$ matrix, a vector in $\mathbb{R}^n$.

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, BA = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 10 \end{bmatrix}.$$

6.4. Example 2 If A is a $n \times n$ matrix and B is a $n \times n$ matrix, then AB is a $n \times n$ matrix again. This is nice because we have now an **algebra**. We can add and multiply $n \times n$ matrices like numbers. But this product is no more commutative. Here is an example

$$A \cdot B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} .$$

$$B \cdot A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} .$$

In general $A \cdot B \neq B \cdot A$.

6.5. Example 3 An other at first a bit strange thing is that there are non-zero (2×2)-matrices A, B such that $A \cdot B = 0$, where $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We can even chose $A = B$. So, here is an example

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

6.6. Example 4 If we take a $1 \times n$ matrix A (a row vector) like $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and a $n \times 1$ matrix B (a column vector) like $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, the product is just what we associate with the dot product: $A \cdot B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 32$. In order to define the dot product between two column vectors $\vec{v}, \vec{w}$, we would therefore define this as the matrix product of $\vec{v}^T \vec{w}$, where $\vec{v}^T$ is the row vector corresponding to $\vec{v}$. We will come to this when we look at inner product spaces.

6.7. Example 5 What happens if we multiply two rotation matrices?

$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin\beta) & -\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin\beta) & \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin)\beta) \end{bmatrix} .$$

We could look up some trig identities. Easier is to think what we get when we rotate by α and then by β: We get a rotation matrix

$$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} .$$

6.8. Example 6 What happens if we multiply to reflection matrices

$$\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(0\alpha) \end{bmatrix} \cdot \begin{bmatrix} \cos(2\beta) & \sin(2\beta) \\ \sin(2\beta) & -\cos(-\beta) \end{bmatrix}$$

$$\begin{bmatrix} \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) & \cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) & \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix} .$$

We can either look up some trig identities or again think geometrically. We get the rotation by $2\alpha - 2\beta$.

$$\begin{bmatrix} \cos(2\alpha-2\beta) & -\sin(2\alpha-2\beta) \\ \sin(2\alpha-2\beta) & \cos(2\alpha-2\beta) \end{bmatrix} .$$

6.9. The concept of a **basis** will allow us later to do matrix multiplication faster. Reminder: A collection of vectors $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis if they span $\mathbb{R}^m$ and if they are linearly independent, meaning that the matrix A with $\mathcal{B}$ in the columns has no free variables after row reduction. To check that m vectors are a basis in $\mathbb{R}^m$, just row reduce the matrix A containing the vectors as columns and see whether it reduces to the **identity matrix**.

$$\boxed{\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\} \text{ is a basis in } \mathbb{R}^m \Leftrightarrow \text{the corresponding matrix } A \text{ row reduces to } 1.}$$