LINEAR ALGEBRA

MATH 21B

Matrix inverse

7.1. The inverse of a linear transformation does not necessarily exist. We first of all need that the matrix is a square $n \times n$ matrix. Then we need to be able to solve $A\vec{x} = \vec{b}$ with a unique $\vec{x}$. We have seen that this is the case if and only if A row reduces to the identity matrix. We call such an A also **invertible**. If we solve $A\vec{x} = \vec{e}_k$, then $A^{-1}\vec{e}_k$ is the k'th column of the inverse matrix. We know how to do this: just write down the augmented matrix $[A|\vec{e}_k]$ and row reduce.

7.2. Instead of doing n row reductions, we can do everything at once by starting with $[A|\vec{e}_1, \dots, \vec{e}_n] = [A|1_n]$ and row reduce this. We will end up with $[1_n|A^{-1}]$. So:

> To find the inverse $B = A^{-1}$ of a $n \times n$ matrix A, row reduce $[A|1_n]$ to $[1|B]$.

7.3. Example: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. To find the inverse, write down

$$[A|1] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

Now row reduce

$$[1|A^{-1}] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] .$$

We have $A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.

7.4. For 2×2 matrices, there is a much faster way to compute this. You "know it". While educators pretend to despise this, "knowing stuff" is the secret behind all modern successful AI. Already the AI pioneer Marvin Minksy quipped ""knowing how to solve a problem is the best and fastest way to solve it". [1]

> If $ad - bc \neq 0$, the inverse of a linear transformation $\vec{x} \mapsto Ax$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is
>
> $$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / (ad - bc) .$$
>
> Magic spell: "Switch diagonals, negate the wings and divide by the determinant"

[1] If you look at modern AI, like Siri, Google, Alexa, ChatGPT or Ernie Bot, they are all are based on knowing large amount of data. They are data slurping monsters who are able to organize them and recombine.

7.5. We know

Theorem 1. *The inverse of an invertible A is unique and $(A^{-1})^{-1} = A$.*

The reason is that if $AB = 1$ and $AC = 1$ then $A(B - C) = 0$ then every column vector of $B - C$ satisfies $A\vec{v} = \vec{0}$. But this means that A has a free variable implying that the columns of A do not form a basis in $\mathbb{R}^n$. So $B - C = 0$ which verifies that $B = C$ is the same.

7.6. Another useful property often used is:

Theorem 2. *If A, B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

We can just check it $AB(AB)^{-1} = ABB^{-1}A^{-1} = AA^{-1} = 1$.

7.7. For some geometric transformations, we can write down the inverse immediately. The inverse of a rotation by an angle α about a line L is a rotation by $-\alpha$ about the line. The inverse of a reflection is itself. A projection onto a line is an example of a transformation that is not invertible.

7.8. Here is a fun way to generate matrices for which the inverse matrix is an integer matrix again. Take a finite set of sets and close it by adding all non-empty subsets. You have now n sets called a simplicial complex. [2] Build the $n \times n$ matrix in which $L_{ij} = 1$ if set i and j intersect and put $L_{ij} = 0$ else. For, example closing $\{\{a,b,c\},\{b,c,d\}\}$ gives $\{\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{b,c,d\}\}$. and the matrix

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, L^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix}$$

7.9. With Mathematica you can plot a matrix using the command "Matrix Plot". An example, Jackson Pollock style:

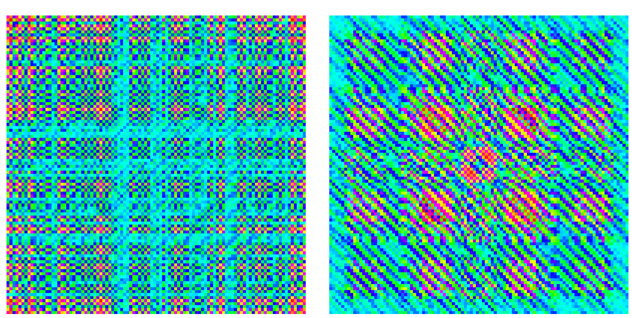

FIGURE 1. The inverse of the 100×100 matrix A defined by $A_{ij} = p_{i+j}$, where p_k is the k'th prime. In the second picture we took $A_{ij} = p_{(1+(i-j)^2)}$.

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[2] O. Knill, The energy of a simplicial complex, Lin alg and Applic, 600, 2020, p. 96-129