LINEAR ALGEBRA

$$\begin{array}{ccc} \vec{v} & \stackrel{S}{\leftarrow} & \vec{c} \\ A\downarrow & & \downarrow B \\ A\vec{v} & \stackrel{S^{-1}}{\rightarrow} & B\vec{c} \end{array}$$

MATH 21B

Coordinates

8.1. A basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ defines an invertible matrix $\boxed{S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}}$.

8.2. If $\vec{v} = c_1\vec{v}_1 + \cdots + c_n v_n$, then c_i are called the **$\mathcal{B}$-coordinates** of $\vec{v}$. We write $\vec{c} = [\vec{v}]_{\mathcal{B}}$ and have $\boxed{S^{-1}\vec{v} = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}}$.

8.3. We can relate the matrix B of the transformation T in the basis $\mathcal{B}$ with the matrix A of T in the standard basis. First use S to get us from the new to the old coordinates, then apply A and finally return to the new coordinates with S^{-1}: this means

$$\boxed{B = S^{-1}AS}\ .$$

8.4. Example: Let T be the reflection at the plane $x + 2y + 3z = 0$. To describe this, use the basis: $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. We have $T(\vec{v}_1) = \vec{v}_1 = [\vec{e}_1]_{\mathcal{B}}, T(\vec{v}_2) = -\vec{v}_2 = [\vec{e}_2]_{\mathcal{B}}, T(\vec{v}_3) = \vec{v}_3 = -[\vec{e}_3]_{\mathcal{B}}$, so that in the new basis the matrix is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We can now get A by computing the inverse of S and multiplying:

$$A = SBS^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 3 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & -2 & -3 \\ 2 & 4 & 6 \\ 3 & 6 & -5 \end{bmatrix} \frac{1}{28} = \begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix} \frac{1}{7}\ .$$

8.5. There are just three important things to remember:

1) The **coordinate change matrix** S has the basis $(\vec{v}_1, \dots, \vec{v}_n)$ as columns.
2) The $\mathcal{B}$ **coordinates** of $\vec{v}$ are $S^{-1}\vec{v}$.
3) The **transformation matrix** in the $\mathcal{B}$ basis is $B = S^{-1}AS$.

8.6. Matrices satisfying $B = S^{-1}AS$ are called **similar**. Similar matrices describe the same transformation. It is just a matter of perspective.

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