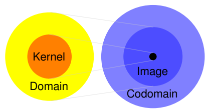

LINEAR ALGEBRA

MATH 21B

Image and Kernel

9.1. The **kernel** of a matrix A is the set of all vectors $\vec{x}$ in the domain such that $A\vec{x} = \vec{0}$. The image of A is the set of all vectors $A\vec{x}$ in the codomain with $\vec{x}$ in the domain.

9.2. For example, if $A = \begin{bmatrix} 5 & 4 & 1 \end{bmatrix}$ then $\ker(A) = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$. This is the plane $5x + 4y + z = 0$. The image of A is the entire codomain $\mathbb{R}$.

9.3. How do we compute the kernel? This is something we have done before. We just need to find the solutions to $A\vec{x} = \vec{0}$.

To find the kernel of A, row reduce A and write down the system of equations at the end. Write the solution in terms of the free variables.

9.4. To find the kernel of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, row reduce A to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. There are two free variables. It is custom to call them with new letters like s, t and rewrite equations, like $x + s + t = 0$, not forgetting $y = s, z = t$. We see that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. A basis for the kernel is $\mathcal{B} = (\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix})$.

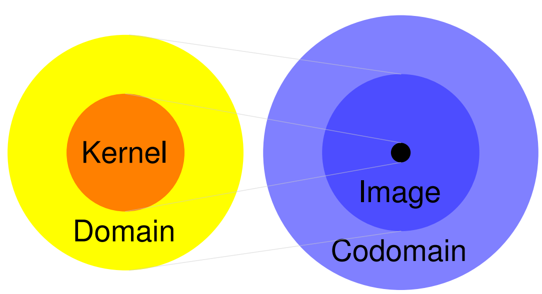

Figure 1. The kernel is part of the domain. The image is part of the codomain.

9.5. As an other example, assume that A is an invertible $n \times n$ matrix. The kernel $\{\vec{x}, A\vec{x} = \vec{0}\}$ consists of only one point $\{\vec{0}\}$ because there is only one solution of $A\vec{x} = \vec{0}$ it is $A^{-1}\vec{0} = \vec{0}$. We say the kernel is **trivial**. The kernel is also called **null space**.

9.6. How do we compute the image? Since all the columns of A are in the image, we already know that the columns span the image. But we do not need all of them. As we can solve any system of equations $A\vec{x} = \vec{b}$ with $\vec{b}$ in the image using free variables put to zero, we see that we only need to look at the leading columns of the original matrix

> To find the image of A, row reduce A and identify the leading variables. The corresponding columns of the original matrix A form a basis for the image of A.

9.7. In the above example with $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, we have seen that the first column is a leading one. The First column vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ therefore spans the image.

9.8. Lets look at the **multiplication table matrix**. Your task is to find the image and kernel of this matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \end{bmatrix} .$$

9.9. Remember the example from the sports team project. You had the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 3 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 3 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 3 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 0 & -1 & 0 & 0 & 3 \end{bmatrix} .$$

This matrix was actually the **Kirchhoff Laplacian** of a graph with two component. There is a two dimensional kernel. We can see this by doing row reduction

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

There are 6 leading variables and so a 2 dimensional kernel. The 6 first columns of A span the image. The kernel can be compute it with the command "NullSpace" in Mathematica.