LINEAR ALGEBRA

MATH 21B

LINEAR SUBSPACES

10.1.

> **Definition:** A subset V of $\mathbb{R}^m$ is called a **linear subspace** of $\mathbb{R}^m$ if it contains $\vec{0}$ and is closed under addition and scalar multiplication. We sometimes just call it a **linear space**.

10.2.

> **Theorem:** $V \subset \mathbb{R}^m$ is a linear subspace of $\mathbb{R}^m$ if and only for all vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^m$ and all real k we have:
>
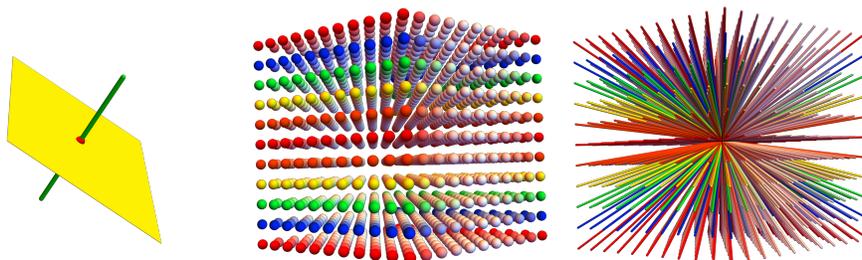> i) $\vec{0} \in V$ ii) $\vec{x} + \vec{y} \in V$ iii) $k\vec{x} \in V$

FIGURE 1. The space $\mathbb{R}^m$, lines, planes or $\vec{0}$ are linear spaces. The union of a plane and a line is not a linear space. The **lattice** $V = \mathbb{Z}^3$ of integer coordinate points or the **hedgehog** consisting of all lines through integer points and 0 are not linear spaces.

10.3. For now we restrict ourselves to linear subspaces of an **ambient space** $\mathbb{R}^m$. Later, we will look at more general **linear spaces**, like the space of all 2×2 matrices or even spaces of functions like the linear space of all polynomials.

10.4. Examples:
1) $\{\vec{0}\}$ is a linear space. It is called the **trivial space** of $\mathbb{R}^m$.
2) $\mathbb{R}^m$ is a linear space in $\mathbb{R}^m$.
3) If A is a $n \times m$ matrix, then $\ker(A)$ is a linear space in $\mathbb{R}^m$.
4) If A is a $m \times p$ matrix, then $\operatorname{im}(A)$ is a linear space in $\mathbb{R}^m$.
5) The set $\{(x, y) \in \mathbb{R}^2, x \geq 0\}$ is not a linear space. Property iii) fails.
6) The graph of $f(x) = 2x$ is a linear space. It is the kernel of $A = [2, -1]$.
7) The graph of $f(x) = 2x + 1$ is not a linear space. It does not contain $\vec{0}$.
8) The space $V^{\perp}$ perpendicular to a linear subspace V is a linear space.

10.5. Some of the following sets are **not** linear subspaces of $\mathbb{R}^3$. Can you see which?

a) The kernel of $A = [1,1,1]$.

b) The image of $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

c) $\{(x,y,z), x,y,z \in \mathbb{Q}\}$.
d) $\{|x| = |y| = |z| \leq 1\}$.

e) The line $x+y=0, y+z=0$.
f) The plane $x+y+z=1$.
g) The unit sphere.
h) The x-axes.
i) $\mathbb{R}^2$ without the point $(1,1)$.
j) The union of all coordinate axis.

10.6. It is important to keep in mind the following fact

Theorem: Image and kernels of a matrix are linear spaces.

10.7.

Definition: A set of vectors $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\} \subset \mathbb{R}^m$ is called **linearly independent** if the matrix $A = \begin{bmatrix} | & \cdots & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & \cdots & | \end{bmatrix}$ has a trivial kernel.

This means that $A\vec{c} = \vec{0}$ has only the trivial solution which means that $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ has only the solution where all $c_j = 0$.

10.8.

Definition: A set of vectors $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is called a **basis** of a linear space V, if $\mathcal{B}$ span V and $\mathcal{B}$ are linearly independent.

10.9. For example, the vectors ($\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$) are a basis of plane they span.

10.10. The vectors ($\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$) are not a basis of $\mathbb{R}^3$. Indeed $\vec{v}_1 + \vec{v}_2 - 4\vec{v}_3 = \vec{0}$ is a **nontrivial linear relation**.

10.11. In $\mathbb{R}^2$, two vectors which are not parallel form a basis.

10.12. In $\mathbb{R}^3$, two vectors never form a basis. Their span is either a plane or (if the two vectors should be parallel) form a line.

10.13. In $\mathbb{R}^3$, three vectors which are not in a common plane form a basis.

10.14. Three vectors in $\mathbb{R}^2$ never form a basis. After row reduction there can only be two leading 1.