LINEAR ALGEBRA

MATH 21B

DIMENSION

11.1.

Definition: The **dimension** of a linear space V is the number $|\mathcal{B}| = k$ of vectors in a basis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_k)$ of V.

This definition is justified because of the **dimension theorem**:

11.2.

Theorem: If $\mathcal{U}, \mathcal{W}$ are both a basis of V, then $|\mathcal{U}| = |\mathcal{W}|$.

It was Ernst Steinitz (1871-1928) (watching you from the paper) who realized that this is not so obvious. He proved an **exchange lemma**: if $\mathcal{U}, \mathcal{W}$ are finite subsets of a linear space such that $\mathcal{U}$ is linearly independent and $\mathcal{W}$ span, then $|\mathcal{U}| \leq |\mathcal{W}|$. This property plays an important role in **matroid theory**. The set of labels of independent column vectors of a matrix is an example of a **matroid**. Impress your friends and mention **matroid oracles**. You will reach "CS guru" status.

11.3.

Definition: The **nullity** of a matrix A is the dimension of the kernel. The **rank** of a matrix A is the dimension of the image.

11.4. The rank nullity theorem is sometimes called the **fundamental theorem of linear algebra**:

Theorem: The sum of the nullity and the rank is the number of columns.

The proof is that after row reduction we either have a leading 1 in a column or no leading 1. The number of leading 1 is the rank. The number of free variables is the nullity. [1]

11.5. If A is a matrix with k columns, then a subset of $\{1, \cdots, k\}$ is called **independent**, if the corresponding set of vectors are independent. The set $\mathcal{M}$ of all such sets is called a **matroid**. If U, V are in $\mathcal{M}$ and U has one more element then V then there is an element $x \in U$ different from V such that $U \cup \{x\}$ is in $\mathcal{I}$. That the set of independent columns of a matrix form a matroid is essentially the exchange lemma. This matroid is the **vector matroid**. An other example of a matroid are subsets of the edge set which are **forests in a graph**. It is called the **graphic matroid**.

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[1]We need that row reduction of a matrix is unique and that dimension is well defined.

11.6. The Rank-Nullity theorem is also known as "fundamental theorem of linear algebra". Isn't it obvious that the number of columns with leading ones plus the number of columns without leading ones are the total number of columns? There is actually a lot packed into that: first of all, one has the fact that **dimension** is a well defined quantity. It also includes the (in linear algebra courses usually assumed and not proven) fact that the row reduction of a matrix is unique [2]

11.7. Let us just mention again an interesting relation between kernel and image. If V is a linear subspace of $\mathbb{R}^m$, then $V^\perp$ is the set of all vectors perpendicular to it. Let A denote the matrix which contains a basis of V as columns. This means $V = \text{im}(A)$. Then A^T, the **transpose matrix**, where the vectors are entered as rows, has the property that the $V^\perp$ is $\ker(A^T)$.

> **Theorem:** $\text{im}(A)^\perp = \ker(A^T)$

11.8. This result is also called **Fredholm alternative**. It links different **dualities**: the **duality** between a matrix A and its transpose A^T, the duality between a subspace V and its orthogonal complement $V^\perp$. This now links image and kernel: **the dual of the image of A is the kernel of the dual of A.**

11.9. This poetry is what makes **mathematicians** excited. It is a general statement, true for all matrices. The duality of kernel and image remains true also in a few thousand years, when you all do not have to worry about linear algebra any more.
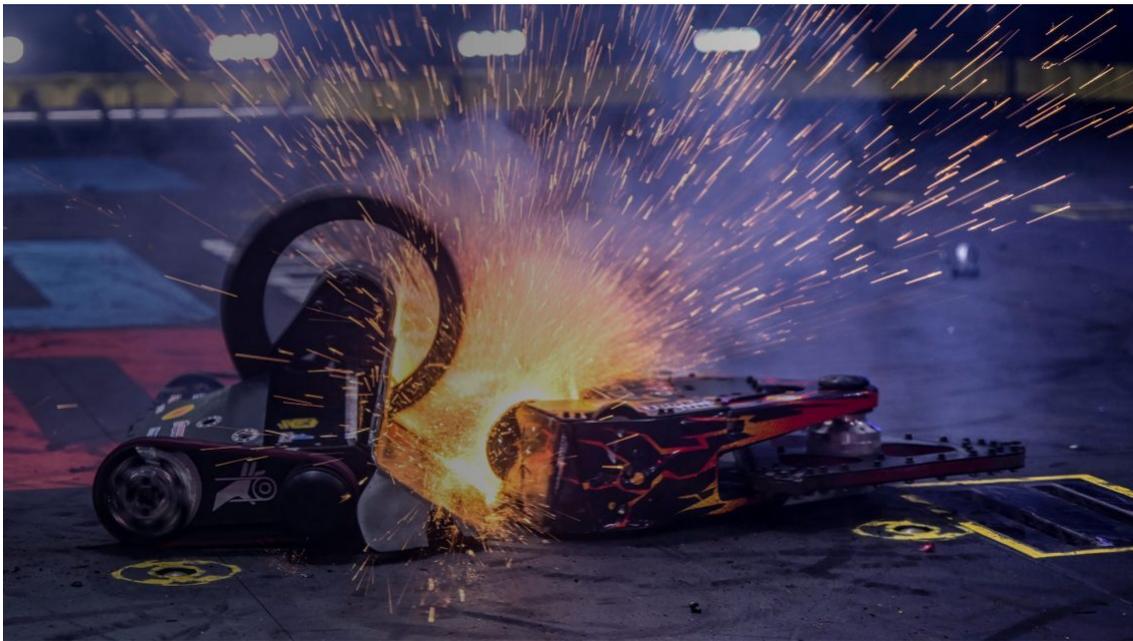

FIGURE 1. The Rank-Nullity theorem will be a theorem even in the far future, when only battle bots will roam the earth. Unlike your worries about doing PSets, mathematics is eternal.

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[2]See https://people.math.harvard.edu/ knill/teaching/math22a2018/handouts/lecture02.pdf